

# A Convection-Driven Dynamo: I. The Weak Field Case

A. M. Soward

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## A CONVECTION-DRIVEN DYNAMO

## I. THE WEAK FIELD CASE

BY A. M. SOWARD

*School of Mathematics, University of Newcastle upon Tyne**(Communicated by Sir Edward Bullard, F.R.S. – Received 18 May 1973)*

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A hydromagnetic dynamo model is considered. A Boussinesq, electrically conducting fluid is confined between two horizontal planes and is heated from below. The system rotates rapidly about the vertical axis with constant angular velocity. It is supposed that instability first sets in as stationary convection characterized by a small horizontal length scale. In preliminary calculations the Lorentz force is neglected so that the magnetic induction equation and the equation of motion are decoupled. The possibility that motions occurring at the onset of instability may sustain magnetic fields is thus reduced to a kinematic dynamo problem. Moreover, the existence of two length scales introduces simplifications which enable the problem to be studied by well-known techniques. The effect of the Lorentz force on the finite amplitude dynamics of the system is investigated also. Since only weak magnetic fields are considered the kinetic energy of the motion is fixed by other considerations and it is only the fine structure of the flow that is influenced by the magnetic field. A set of nonlinear equations, which govern the evolution of the hydromagnetic dynamo, are derived from an asymptotic analysis. The equations are investigated in detail both analytically and numerically. In spite of serious doubts concerning the existence of sufficiently complex stable motions, stable periodic dynamos are shown to exist. An interesting analytic solution of these equations, which may be pertinent to other problems arising from finite-amplitude Bénard convection, is presented in the final section.

## 1. INTRODUCTION

Progress towards the understanding of kinematic dynamos has advanced rapidly in recent years (see, for example, Roberts 1971). Indeed a wide variety of prescribed fluid flows are now known to maintain magnetic fields by dynamo action. It is natural, therefore, to continue by investigating the hydromagnetic dynamo problem in which the magnetic induction equation and the equation of motion are attacked simultaneously. In other words, the velocity distribution is no

longer prescribed, as in the kinematic models, but is determined from the equation of motion which itself is influenced by the induced magnetic field via the Lorentz force. Thus the two equations are completely coupled.

Busse (1973) has initiated† an approach to the hydromagnetic dynamo problem which is based on perturbations to the solutions of the linear equations governing non-magnetic Bénard convection. He has shown that steady rolls occurring at the onset of instability, when combined with a shear flow along the axis of the rolls, are capable of sustaining a magnetic field provided motion is sufficiently vigorous. This result suffers from the shortcomings of any kinematic analysis, namely it is possible to obtain solutions for the magnetic field which grow indefinitely. But, if it is anticipated that in the full hydromagnetic dynamo the magnetic field is steady, the amplitude of the motion characterized here by the magnetic Reynolds number is fixed by the kinematic dynamo problem! Busse supposes that in the absence of Lorentz forces, an equilibrium amplitude for the rolls is determined by equating the small non-linear convective terms with buoyancy forces which result from a small deviation of the Rayleigh number above its critical value. When the Lorentz force is included the result is modified. However, since the magnitude of the velocity is already known from consideration of the magnetic induction equation, the equilibrium condition fixes the magnitude of the Lorentz force. In this way the hydromagnetic dynamo is solved: both the amplitude of the motion and the magnetic field are determined.

The hydromagnetic model considered here has similarities with that of Busse (1973): a critical comparison is given in appendix A. A layer of electrically conducting Boussinesq fluid is confined between horizontal, stress free, perfectly conducting planes distance  $L$  apart. The system rotates with constant angular velocity  $\Omega$  about the vertical axis and the Taylor number

$$T = 4\Omega^2 L^4 / \nu^2, \quad (1.1)$$

where  $\nu$ , the kinematic viscosity, is assumed to be large. A temperature difference,  $\Delta T$ , is maintained between the planes which are kept at constant temperatures. It is assumed that the Rayleigh number

$$R = \gamma g L^3 \Delta T / \nu \kappa, \quad (1.2)$$

where  $\gamma$  is the coefficient of thermal expansion,  $\kappa$  is the thermal diffusivity,  $g$  is the acceleration due to gravity is close to the critical value  $R_0(k)$  which corresponds to steady motion with horizontal wave vector  $k$ . To ensure that instability first sets in as stationary convection it is supposed that the Prandtl number

$$\sigma = \nu / \kappa \quad (1.3)$$

is greater than unity. Chandrasekhar (1961) has shown that this condition is sufficient to exclude overstability. When the fluid is in motion a magnetic field  $\mathbf{b}^*$  permeates the fluid. It is supposed that there are no external electric currents so that magnetic field is maintained against ohmic decay solely by the induction process. Under certain conditions to be specified later, the magnetic field and flow remain sufficiently small for the linear stability analysis of the non-magnetic problem to remain valid for all time. Since the solution of the linear problem is not unique, higher order approximations must be considered which take the nonlinear terms into account. The Lorentz force becomes important at this stage and, since it plays such a vital role in determining the motion of the fluid, the dynamo is truly hydromagnetic.

† There is, of course, previous work on the hydromagnetic dynamo some of which is discussed by Busse (1973) His work is, however, particularly pertinent to the analysis of this paper.

It is well known that, when the Taylor number is large, the most unstable modes for the non-magnetic problem have horizontal wavenumbers of order  $T^{\frac{1}{2}}L^{-1}$ . Consequently the horizontal length scale of the motion at the onset of instability is of order  $T^{-\frac{1}{2}}L$ , while the corresponding Rayleigh number is order  $T^{\frac{3}{2}}$ . The resulting kinematic† dynamo is thus characterized by two length scales  $L$  and  $T^{-\frac{1}{2}}L$  and is consequently amenable to well-known multiple length scale procedures which are an extension of those described by Childress (1967, 1970). The magnetic field may be divided into two parts. First, there is the mean horizontal magnetic field which is dependent only on the vertical coordinate. Secondly, there is a smaller contribution which fluctuates on the horizontal length scale  $T^{-\frac{1}{2}}L$ . The essence of the dynamo mechanism is that the two contributions are mutually supporting: convection of the mean magnetic field induces the smaller fluctuating magnetic field, while convection of the fluctuating field helps to support the mean field.

The mathematical procedure used, in the limit  $T \rightarrow \infty$ , is straightforward. Once it is established that the motion occurring at the onset of convection can sustain a magnetic field it only remains to determine the evolution of the velocity distribution and magnetic field by taking account of the Lorentz force and other small nonlinear effects. The analysis is simplified by the assumptions

$$\sigma = O(1) \quad \text{and} \quad \sigma_{\eta} = \eta/\kappa = O(1), \quad (1.4)$$

where  $\eta$  is the magnetic diffusivity. Thus the order of magnitude of all three diffusion time scales  $L^2/\eta$ ,  $L^2/\kappa$ ,  $L^2/\nu$  is the same and all expansions can be based on the single parameter

$$\epsilon = T^{-\frac{1}{2}} \quad (\ll 1). \quad (1.5)$$

It is found that, when

$$0 < R - R_0(k_c) = O(T^{\frac{1}{2}}), \quad (1.6)$$

where the horizontal wavenumber,  $k_c$ , is chosen to minimize  $R_0(k)$ , finite amplitude motion is maintained which can support a magnetic field by the dynamo process described above. If the fluid velocity  $\mathbf{u}^*$  and the magnetic field  $\mathbf{b}^*$  are made dimensionless by the substitutions

$$\mathbf{u}^* = (\eta/L) \epsilon^{-\frac{1}{2}} \mathbf{u}, \quad \mathbf{b}^* = \mathcal{B} \mathbf{b}, \quad (1.7)$$

the finite amplitude motions are characterized by  $\mathbf{u}$  and  $\mathbf{b}$  of order 1. The strength of the magnetic field can be measured by the Hartmann number

$$M = [\mathcal{B}^2 L^2 / \mu \rho \nu \eta]^{\frac{1}{2}}, \quad (1.8)$$

and is determined by solving the hydromagnetic dynamo problem. Since the magnitude of the field depends on the vigour of the motion, which in turn depends on the strength of the buoyancy forces, the value of  $M$  ultimately depends on the choice of Rayleigh number. It transpires that for the parameter range (1.6) solutions of the hydromagnetic dynamo can be found having weak magnetic fields for which  $M$  is order 1. This régime was described by Childress & Soward (1972) as the weak-field limit. Investigation into other régimes is concurrently in progress by Childress.

For the parameter range discussed in the previous paragraph advection of the magnetic field proceeds at the same rate as its diffusion. Consequently the mean magnetic field evolves on the diffusion time scale. The time dependence of the flow is more complicated. In order to isolate the

† Whenever the velocity is assumed to be uninfluenced by the magnetic field the dynamo is referred to as kinematic.

dynamo mechanism, solutions are considered for which the flow evolves on the diffusion time scale. Of particular interest is the response of the flow to changes in magnitude of the Lorentz force. However, it is possible that the flow could vary on much faster time scales. Indeed a natural **time** scale arises whenever the time derivatives in the equation of motion are assumed to be comparable in magnitude with the remaining terms. For the hydromagnetic dynamo solutions considered here, the time derivatives are negligible in the low-order approximations and hence the solutions may be regarded as quasi-steady. The validity of solutions based on a single time scale is a delicate matter discussed in detail in §2.

The governing equations and boundary conditions defining the hydromagnetic dynamo are described in §2. Series solutions of the equations are developed with the use of  $\epsilon^{\frac{1}{2}}$  as an expansion parameter. The zeroth order solutions of the non-magnetic problem are derived in §3. Though the results of the stability problem are well known (see Chandrasekhar 1961), a new derivation is given since it provides the basis for the subsequent determination of the higher order terms in the expansion of the solution. The lack of vertical boundaries implies that only the magnitude of the horizontal wave vector is determined from the zero-order analysis; its direction is arbitrary. Thus a continuum of modes with  $|\mathbf{k}| = \text{constant}$  is possible while the dependence of each mode on the slow diffusion time is unknown. Consequently the motion is undetermined at this stage and it is necessary to consider higher order terms in the expansion in order to specify uniquely the flow, as defined by the zeroth-order solution. The subsequent analysis, can, however, be simplified by restricting attention to particular modes as the procedure is consistent with the governing equations. This, of course, does not provide a complete treatment of the stability problem and no such claim is made.

The kinematic dynamo is investigated for arbitrary steady flows compatible with the zeroth-order solutions. The analysis is based on equation (3.21) which governs the development of the mean magnetic field. The term involving  $M_{ij} B_j$  provides a source of magnetic field which hopefully overcomes the ohmic dissipation  $-\partial^2 B_i / \partial z^2$ : In the language of mean field electrodynamics it is more commonly called the  $\alpha$ -effect (Steenbeck & Krause 1969). Provided the motion is sufficiently vigorous, dynamo action is generally possible. One notable exception is the case of a single roll characterized by a single value of  $\mathbf{k}$ : for this mode dynamo action is always impossible. Now it transpires that the average kinetic energy per unit volume

$$\mathcal{T}^* = (\rho\eta^2/\epsilon L^2)\mathcal{T}, \quad (1.9)$$

is a constant to lowest order. In fact, the dimensionless quantity  $\mathcal{T}$  is fixed by the difference between the Rayleigh number  $R$  and its critical value  $R_0(k)$  so that, when  $R = R_0(k)$ , there is no motion ( $\mathcal{T} = 0$ ), while, as  $R - R_0(k)$  increases,  $\mathcal{T}$  increases monotonically also. Consequently the Rayleigh number not only provides an important constraint on the type of motion possible, but also provides an unambiguous measure of the magnetic Reynolds number. Moreover, it is found that a critical value of  $\mathcal{T}/R_0(k)$  ( $= \Lambda$ ) exists for which all solutions of the magnetic induction equation decay when  $\Lambda < \Lambda_{\text{crit}}$ , while a magnetic field which oscillates with time is maintained when  $\Lambda = \Lambda_{\text{crit}}$ . Thus, when the response of the fluid motion to the time-dependent Lorentz force is taken into account, motions varying on the slow time scale must be expected. This observation highlights the deficiency of any study of the kinematic dynamo based on steady motions. Since the general problem involving unsteady flow is formidable, a particularly simple model is investigated which isolates important new effects. Further it provides the prototype of a general class of motions which arise when the hydromagnetic dynamo is considered later. The

most important conclusion to be drawn from the model is that unsteady flow can provide more efficient dynamo action than steady flow. In other words, kinematic dynamos are possible with  $\Lambda < \Lambda_{\text{crit}}$ . Whether this is true for hydromagnetic dynamos cannot, of course, be decided on the basis of the kinematic analysis. However, for each class of motion used to investigate hydro-magnetic dynamos, solutions are found for which  $\Lambda < \Lambda_{\text{crit}}$ , as suggested by the kinematic analysis.

The higher order terms in the expansion of the velocity and temperature distributions are determined in §§4 and 5 for the sole purpose of removing the indeterminacy in the zeroth-order solution. Now, since the motion is the superposition of various rolls, its structure is defined uniquely by specification of  $\hat{w}^{(0)}(t, \mathbf{k})$ , which defines the vertical velocity

$$\text{Re} [\hat{w}^{(0)}(t, \mathbf{k}) \sin \pi z e^{i\mathbf{k} \cdot \mathbf{x}}]$$

for each roll;  $Lz$  is the vertical distance from the bottom plane,  $L^2 t / \eta$  is the time. The culmination of the asymptotic analysis is the determination of (5.12); a set of equations which governs the evolution of each roll separately. The main effects isolated by this equation are the nonlinear interactions and the Lorentz forces represented by

$$\sum_{\mathbf{k}'} A(\mathbf{k}', \mathbf{k}) |\hat{w}^{(0)}(t, \mathbf{k}')|^2 \hat{w}^{(0)}(t, \mathbf{k}) \quad \text{and} \quad -\frac{1}{2} m(t, \mathbf{k}) \hat{w}^{(0)}(t, \mathbf{k})$$

respectively. Here  $m(t, \mathbf{k})$  is a weighted vertical average of  $(\mathbf{k} \cdot \mathbf{B})^2$ . The additional unknown  $\hat{\Theta}^{(2)}(t)$ , which measures a second-order correction to the horizontal average of the temperature, is eliminated by the condition that the kinetic energy is constant. The constraint on the kinetic energy is a manifestation of the result that the lowest order horizontal average of the temperature, as measured by  $\hat{\Theta}^{(0)} \sin 2\pi z$ , is independent of time and is fixed by the value of the Rayleigh number. It follows that a more useful form of (5.12) is (7.1), in which  $\hat{\Theta}^{(2)}(t)$  has been eliminated and  $q(t, \mathbf{k}) = |\hat{w}^{(0)}(t, \mathbf{k})|^2$ . Some additional effects are accounted for by the term involving  $r(\mathbf{k})$ . The term is less important than the others. Indeed when  $|\mathbf{k}|$  is the same for all modes, as in the case when only the most unstable modes on linear theory are investigated, the term is identically zero. The evolution of the hydromagnetic dynamo is, therefore, determined by (7.1) and the dynamo equation (3.21) governing the mean horizontal magnetic field.

Investigations into finite amplitude Bénard convection usually indicate, as for example the case of the non-rotating system considered by Schluter, Lortz & Busse (1965), that certain convective rolls are the only stable motion. In other words, ultimately as  $t \rightarrow \infty$ , the velocity distribution is likely to approach that corresponding to a single roll. Until calculations are actually performed it is not known whether corresponding results hold for the rotating layer in the presence of a non-uniform magnetic field but if they do the dynamo must ultimately fail as the motion is not sufficiently asymmetric. The analysis of the hydromagnetic dynamo equation (3.21) and (7.1) is a matter of some complexity and a detailed examination of the properties of the equations is made in §§7 and 8. The preliminary investigation of dynamo models based on motions which are represented by the superposition of two rolls intersecting at right angles and three rolls intersecting each other at an angle of  $\frac{1}{3}\pi$  is undertaken in §7. When there is no magnetic field and only the most readily excited modes for which  $k = k_c$  are considered, all terms except the first two in (7.1) are non-zero. Hence the amplitudes of the modes  $\hat{w}^{(0)}(t, \mathbf{k})$  evolve as a direct consequence of the non-linear interactions. For the case of the two rolls the flow is steady while for three rolls non-linear periodic oscillations ensue defined by (7.25) to (7.29). The result, which is apparently new, emphasizes the danger of general conclusions concerning the ultimate

state of Bénard convection based on the linear stability analysis of steady finite amplitude motion. For with the exception of a few special cases, it may be anticipated that finite amplitude flows are generally unsteady. Such periodic motions, some of which are possibly stable to yet further perturbations, are necessarily overlooked when steady motion is investigated. Analytic solutions of the hydromagnetic dynamo are not obtained for either the case of two or three rolls. Instead, numerical integration of initial value problems indicate the existence of stable†, nonlinear, periodic dynamos. These periodic solutions are approached as  $t \rightarrow \infty$  and provide the limit cycles for the initial value problems investigated. The case of two rolls represented by modes of slightly different wavenumber is easily seen to be unstable in the absence of magnetic field (see (7.5) and (7.7)). However, the numerical integrations show that the dynamo is stabilized by the presence of the magnetic field! The numerical calculations for three rolls leads on naturally to the analytic solution presented in §8. Here, instead of restricting attention to motion expressed by the superposition of a set of discrete rolls, a continuous distribution is envisaged. Paradoxically, it is this added complexity which permits analytic treatment of a class of periodic solutions. Since hydro-magnetic dynamos are obtained for the continuous case which correspond to Rayleigh numbers smaller than any found for the case of two or three rolls, it is reasonable to suppose that these periodic dynamos are the limit cycles resulting from a wide class of initial conditions. Indeed it seems likely that of the cases considered, the continuous distribution is the most pertinent. It may also be noted that the same technique can be applied to the classic (non-magnetic) situations and provide simple analytic solutions, describing periodic motion, which have apparently been overlooked until now.

It should be noted that the results of the paper, with the exceptions of §§ 7 *b*, *c* and 8 have already been reported by Childress & Soward (1972) in a preliminary and highly condensed form. One objective of the present paper is to give a detailed account of the asymptotic analysis which leads to equations (3.21) and (7.1) governing the hydromagnetic dynamo in the so-called weak-field limit. The second objective is to describe analytic and numerical calculations which indicate the existence of stable oscillatory solutions of these nonlinear equations.

## 2. THE GOVERNING EQUATIONS

Rectangular cartesian coordinates  $x^*$ ,  $y^*$ ,  $z^*$  are adopted where the  $z^*$ -direction is vertical and the  $x^*$ - and  $y^*$ -directions lie in the horizontal plane. Since the planes are distance  $L$  apart and the horizontal length scales of interest are order  $\epsilon L$ , the position vector  $\mathbf{x}^*$  is made non-dimensional by the substitution

$$\mathbf{x}^* = (\epsilon Lx, \epsilon Ly, Lz). \quad (2.1)$$

Moreover, since the dynamo process is concerned with events occurring on the magnetic diffusion time scale,  $L^2/\eta$  is adopted as the unit of time. Another quantity, as yet undefined, is the temperature

$$T^* = T_0^* + \Delta T\{-z + \epsilon\theta\}. \quad (2.2)$$

Here  $T_0^*$  is the temperature on the bottom plane  $z = 0$  and the dimensionless variable  $\theta$  measures the perturbation of the temperature from its undisturbed linear profile.

† Since the stability of the solution to arbitrary perturbations is not considered, 'stable' is used in a limited sense. It refers to the possibility that the dynamo can be maintained by a certain class of motions without it ultimately failing, as  $t \rightarrow \infty$ .

It is convenient to separate the variables  $\mathbf{u}$ ,  $\mathbf{b}$ , and  $\theta$  into two parts. The first part is the horizontal average which depends only on  $z$  and time  $t$ . The second part is the remainder, labelled with a prime, which has zero horizontal average denoted by  $\langle \dots \rangle$ . Thus one defines

$$\mathbf{u} = \mathbf{u}'(\mathbf{x}_h, z, t) + \epsilon^3 \mathbf{U}_h(z, t), \quad (2.3a)$$

$$\mathbf{b} = \mathbf{B}_h(z, t) + \epsilon^{\frac{1}{2}} \mathbf{b}'(\mathbf{x}_h, z, t), \quad (2.3b)$$

$$\theta = \Theta(z, t) + \epsilon^{\frac{1}{2}} \theta'(\mathbf{x}_h, z, t), \quad (2.3c)$$

where quantities are scaled so that they are order 1 and the suffix  $h$  denotes the horizontal components of the vector.

The above substitutions are made in the governing equations and, for brevity, the prime is suppressed. Hence the horizontal and vertical components of the equation of motion are

$$\begin{aligned} \frac{\eta}{\nu} \left\{ \epsilon^3 \frac{\partial \mathbf{u}_h}{\partial t} + \epsilon^{\frac{3}{2}} \mathbf{u}_h \cdot \nabla \mathbf{u}_h + \epsilon^{\frac{5}{2}} w \frac{\partial \mathbf{u}_h}{\partial z} \right\} + \mathbf{i}_z \times (\mathbf{u}_h + \epsilon^3 \mathbf{U}_h) \\ = -\nabla_h p + M^2 \epsilon^3 (\mathbf{B}_h \cdot \nabla) \mathbf{b}_h + \left( \epsilon \nabla_h^2 + \epsilon^3 \frac{\partial^2}{\partial z^2} \right) \mathbf{u}_h, \end{aligned} \quad (2.4a)$$

and

$$\begin{aligned} \frac{\eta}{\nu} \left\{ \epsilon^3 \frac{\partial w}{\partial t} + \epsilon^{\frac{3}{2}} \mathbf{u}_h \cdot \nabla_h w + \epsilon^{\frac{5}{2}} w \frac{\partial w}{\partial z} \right\} \\ = -\frac{\partial}{\partial z} (P + p) + \sigma_\eta^{-1} \{ \epsilon^{\frac{1}{2}} \tilde{R} \Theta + \epsilon \tilde{R} \theta \} + M^2 \epsilon^3 (\mathbf{B}_h \cdot \nabla) b_z + \left( \epsilon \nabla^2 + \epsilon^3 \frac{\partial^2}{\partial z^2} \right) w, \end{aligned} \quad (2.4b)$$

respectively, where

$$R = T^{\frac{3}{2}} \tilde{R}. \quad (2.5)$$

The vertical velocity is  $w$  and only terms up to order  $\epsilon^3$  have been retained. The operator  $\nabla_h$  denotes the horizontal gradient and  $P(z, t) + p(\mathbf{x}_h, z, t)$  is the dimensionless form of the modified pressure. Similarly, the heat conduction equation is separated into its mean part

$$\sigma_\eta \left\{ \frac{\partial \Theta}{\partial t} + \frac{\partial}{\partial z} \langle w \theta \rangle \right\} = \frac{\partial^2 \Theta}{\partial z^2}, \quad (2.6a)$$

and its fluctuating part

$$\sigma_\eta \left\{ \epsilon^2 \frac{\partial \theta}{\partial t} - w + \epsilon w \frac{\partial \Theta}{\partial z} + \epsilon^{\frac{1}{2}} \left[ \mathbf{u}_h \cdot \nabla \theta + \epsilon w \frac{\partial \theta}{\partial z} - \epsilon \frac{\partial}{\partial z} \langle w \theta \rangle \right] \right\} = \left[ \nabla_h^2 + \epsilon^2 \frac{\partial^2}{\partial z^2} \right] \theta. \quad (2.6b)$$

The mean part of the magnetic field  $\mathbf{B}_h(z, t)$  satisfies the equation

$$\frac{\partial \mathbf{B}_h}{\partial t} = \frac{\partial}{\partial z} \{ \mathbf{i}_z \times \langle \mathbf{u} \times \mathbf{b} \rangle \} + \frac{\partial^2 \mathbf{B}_h}{\partial z^2}, \quad (2.7a)$$

and to lowest order the fluctuating magnetic field  $\mathbf{b}$  is determined from the equation

$$0 = (\mathbf{B}_h \cdot \nabla) \mathbf{u} + \nabla_h^2 \mathbf{b}. \quad (2.7b)$$

The remaining equations governing  $\mathbf{u}$  and  $\mathbf{b}$  are

$$\epsilon (\partial w / \partial z) = -\nabla \cdot \mathbf{u}_h \quad \text{and} \quad \epsilon (\partial b_z / \partial z) = -\nabla \cdot \mathbf{b}_h. \quad (2.8)$$

When  $\mathbf{u}$  is given, equations (2.7) are similar to the well-studied multiple length-scale dynamo equations considered by Childress (1967). They differ only in as much as there is no small length scale in the  $z$  direction. The added complication implies that, even when the velocity is steady, equation (2.7a) cannot be reduced to an equation with constant coefficients. Instead, a term  $\sin 2\pi z$  appears in equation (3.21) governing the mean magnetic field  $\mathbf{B}_h(z, t)$  which, in turn, is



represented conveniently by a Fourier series in  $z$ . Hence for steady motion, the kinematic dynamo problem is reduced to determining the eigenvalues (the growth rate of the magnetic field) and corresponding eigenvectors (defining the coefficients of the Fourier series) of a complex tri-diagonal matrix which are readily obtained by numerical computation.

For simplicity it is supposed that the horizontal boundaries are stress free, perfectly conducting and isothermal. The boundary conditions for the mathematical problem are then just

$$\left. \begin{aligned} w &= \partial^2 w / \partial z^2 = 0, \\ b_z &= \partial^2 b_z / \partial z^2 = 0, \quad \partial \mathbf{B}_h / \partial z = 0, \\ \theta &= \Theta = 0. \end{aligned} \right\} \text{ on } z = 0, 1. \quad (2.9)$$

It may be anticipated that more realistic boundary conditions will not substantially alter the results.†

It is assumed that all dimensionless parameters are order 1 with the exception of  $\epsilon$  which is small,

$$\epsilon \ll 1. \quad (2.10)$$

Though the assumptions may not be physically realistic (e.g.  $\sigma_\eta \sim 10^5$  in the fluid core of the Earth) they have the mathematical advantage of restricting the solution to a single expansion parameter. Consequently, the physical mechanisms are not obscured by multiple expansions. Solutions are sought in powers of  $\epsilon^{\frac{1}{2}}$  (or  $T^{-\frac{1}{2}}$ ) so that, for example, the non-dimensional temperature  $\theta$  (see equation (2.2)) is written in the form

$$\theta = \{\Theta^{(0)}(z, t) + \epsilon^{\frac{1}{2}}\Theta^{(1)}(z, t) + \dots\} + \epsilon^{\frac{1}{2}}\{\theta^{(0)}(\mathbf{x}_h, z, t) + \epsilon^{\frac{1}{2}}\theta^{(1)}(\mathbf{x}_h, z, t) + \dots\}. \quad (2.11)$$

Similar expansions are adopted for the velocity and magnetic field vectors. The series representations are substituted into (2.4) to (2.8). The coefficients  $\epsilon^{\frac{1}{2}n}$ , where  $n$  is an integer, are equated and this leads to a hierarchy of equations which are solved systematically. Evidently the equation of motion must be considered up to sixth order before the Lorentz force has any influence! However, none of the important unknowns such as the velocity  $\mathbf{u}^{(0)}$  or the magnetic field  $\mathbf{B}_h^{(0)}$  are determined uniquely until this point is reached.

The whole expansion procedure hinges on the correct choice of Rayleigh number which is assumed to be approximately  $\tilde{R}^{(0)}(k)$ ; the critical value for disturbances with wavenumber  $k$ . Most of the analysis is concerned with the case  $k = k_c$ , which makes  $\tilde{R}^{(0)}(k)$  a minimum, since this gives the lowest value of  $R$  for which convection can occur. It is supposed that  $\tilde{R}$  has the expansion

$$\tilde{R} = \tilde{R}^{(0)}(k) + \epsilon \tilde{R}^{(2)}(k) + \epsilon^2 \tilde{R}^{(4)}(k). \quad (2.12)$$

The value of  $\tilde{R}^{(2)}$  is the key parameter as it determines the convective heat transport and hence the kinetic energy of the motion. Evidently this measure of the flow is fundamental to the dynamo mechanism since it also characterizes the strength of the induction process. The inclusion of an  $\epsilon^2$  term at first sight appears unnecessary: But suppose the motion is made up of modes of wavenumber  $k$ , all differing slightly from some fixed value. Then the linear problem for each mode

† If, instead of the stress-free condition, the no-slip condition is applied at the boundaries, the formation of Ekman layers will have a controlling influence on the interior flow. The vertical motion induced by Ekman layer suction will have the order of magnitude of the horizontal velocity modified by the factor  $T^{-\frac{1}{2}}\epsilon^{-1}$  ( $= \epsilon^{\frac{1}{2}}$ ) and to lowest order  $w^{(1)} = -\frac{1}{2}\nabla_h^2 p^{(0)}$  ( $= -\frac{1}{2}k^2 p^{(0)}$ ) is the boundary condition to be applied to the interior flow at  $z = 0$ . Though the results of the stability problem are not affected to lowest order, some modifications are required in the subsequent analysis.

would involve different values of  $\tilde{R}^{(0)}(k)$ . Now the analysis implies that  $\tilde{R}^{(2)}(k)$  must be the same for all modes (see (4.10), (4.11)). Consequently, provided  $\tilde{R} - \tilde{R}^{(0)}(k) - \epsilon \tilde{R}^{(2)}(k)$  is order  $\epsilon^2$ , it may be used as the definition of  $\tilde{R}^{(4)}(k)$ . Thus the  $\epsilon^2$  term allows a little flexibility in choice of wavenumbers for the basic disturbance (see figure 1).

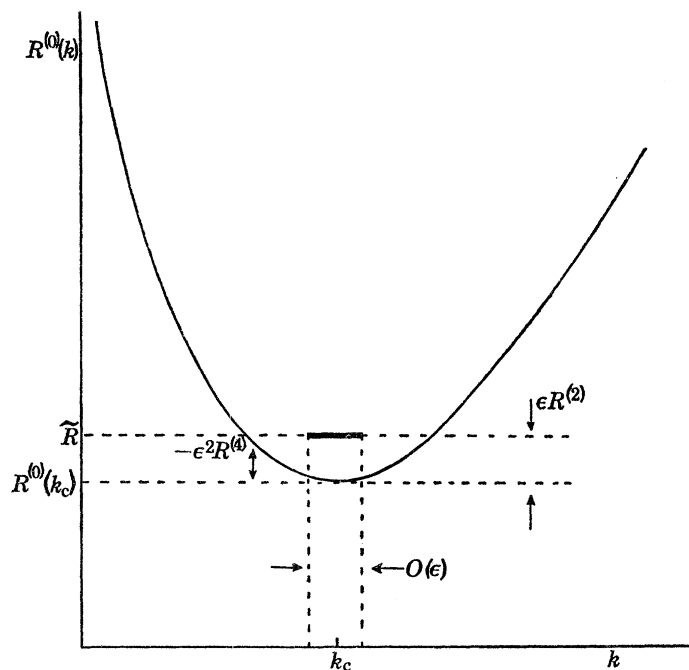


FIGURE 1. A schematic drawing of the neutral stability curve  $R^{(0)} = k^4 + \pi^2/k^2$ . In the calculations  $\tilde{R}$  is given and the wavenumbers considered are restricted by the condition that  $k - k_c = O(\epsilon)$ .

Finally the assumption, that the time scale is order 1, is a filtering approximation. It is the technique usually adopted in geophysics and related subjects to filter out high-frequency modes which are believed to be unimportant with regard to the general development of the system. The procedure is, of course, consistent with the equations, but prohibits the consideration of arbitrary initial value problems. The advantage of the approximation in the present context is that it isolates the temporal development to the single diffusion time scale over which the dynamo evolves. However, a fast time scale order  $\epsilon^{1/2}$  is introduced briefly in § 6 when the stability of the motion to order  $\epsilon^{1/2}$  perturbations is considered. Though these are just the type of motions the filtering approximation rules out, it is encouraging to find that they decay on the diffusion time scale in any case.

### 3. THE EIGENVALUE PROBLEM AND THE MEAN FIELD EQUATIONS

The linear eigenvalue problem for  $\tilde{R}^{(0)}(k)$  is developed within the framework of the expansion procedures outlined in the previous section. Subsequent evaluation of higher order terms is performed in a similar manner except that there are additional forcing terms resulting from nonlinear interactions. Thus the simple calculations leading to the well-known equations (3.11) and (3.12), given in their more general form by Chandrasekhar (1961), form the basis for much of the analysis in §§ 3 to 5.

To lowest order the flow is geostrophic and the horizontal velocity satisfies the equations

$$\mathbf{i}_z \times \mathbf{u}_h^{(0)} = -\nabla_h p^{(0)}, \quad (3.1a)$$

$$\nabla \cdot \mathbf{u}_h^{(0)} = 0. \quad (3.1b)$$

Equations (3.1) have the solution

$$\mathbf{u}_h^{(0)} = \mathbf{i}_z \times \nabla p^{(0)}. \quad (3.2)$$

In order to determine  $\mathbf{u}_h^{(0)}$ , the horizontal components of the equation of motion must be considered up to order  $\epsilon$ , giving

$$\mathbf{i}_z \times \mathbf{u}_h^{(2)} = -\nabla_h p^{(2)} + \nabla_h^2 \mathbf{u}_h^{(0)}. \quad (3.3)$$

With the help of (3.2) the solution of this equation is

$$\mathbf{u}_h^{(2)} = \mathbf{i}_z \times \nabla p^{(2)} + \nabla_h (\nabla_h^2 p^{(0)}). \quad (3.4)$$

Since the continuity equation (2.8) gives

$$\partial w^{(0)} / \partial z = -\nabla \cdot \mathbf{u}_h^{(2)}, \quad (3.5)$$

the divergence of (3.4) leads to the equation

$$\partial w^{(0)} / \partial z = -\nabla_h^4 p^{(0)} = -k^4 p^{(0)}, \quad (3.6)$$

where it is assumed that

$$\nabla_h^2 p^{(0)} = -k^2 p^{(0)}. \quad (3.7)$$

The problem for  $\mathbf{u}_h^{(0)}$  is completed by considering the order  $\epsilon$  terms of the vertical component of the equation of motion (2.4*b*) and the order 1 terms of the heat conduction equation (2.6*b*). This gives, respectively,

$$0 = -\partial p^{(0)} / \partial z + \sigma_\eta^{-1} \tilde{R}^{(0)} \theta^{(0)} + \nabla_h^2 w^{(0)}, \quad (3.8)$$

and

$$-\sigma_\eta w^{(0)} = \nabla_h^2 \theta^{(0)}. \quad (3.9)$$

Equations (3.6) to (3.9) may now be reduced to the single equation

$$\partial^2 w^{(0)} / \partial z^2 + (\tilde{R}^{(0)} k^2 - k^6) w^{(0)} = 0. \quad (3.10)$$

Attention is restricted to the solution

$$w^{(0)}(\mathbf{x}_h, z, t) = \bar{w}^{(0)}(\mathbf{x}_h, t) \sin \pi z, \quad (3.11a)$$

$$p^{(0)}(\mathbf{x}_h, z, t) = -\frac{\pi}{k^4} \bar{w}^{(0)}(\mathbf{x}_h, t) \cos \pi z, \quad (3.11b)$$

$$\theta^{(0)}(\mathbf{x}_h, z, t) = \frac{\sigma_\eta}{k^2} \bar{w}^{(0)}(\mathbf{x}_h, t) \sin \pi z, \quad (3.11c)$$

which satisfies all the boundary conditions (2.9) and where

$$\tilde{R}^{(0)}(k) = k^4 + (\pi/k)^2. \quad (3.11d)$$

Of particular interest is the value

$$k_c = \pi^{1/3} / 2^{2/3}, \quad (3.12a)$$

which defines the minimum value of  $\tilde{R}^{(0)}(k)$ , namely

$$\tilde{R}^{(0)}(k_c) = 3\pi^{1/3} / 2^{2/3}. \quad (3.12b)$$

Indeed it is this result which dictates the choice of horizontal length scale  $\epsilon L$ . At this stage it is convenient to introduce the Fourier representation

$$\bar{w}^{(0)}(\mathbf{x}_h, t) = \sum_{|\mathbf{k}|=k} \hat{w}^{(0)}(t, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (3.13)$$

where  $\hat{w}^{(0)}(t, \mathbf{k}) = \hat{w}^{(0)*}(t, -\mathbf{k})$  (here the star denotes complex conjugate) and summation is over horizontal wavenumbers  $\mathbf{k}$ . Since there is no restriction on the values of  $\hat{w}^{(0)}(t, \mathbf{k})$  the solution is not unique. Indeed not only is the value of  $\hat{w}^{(0)}(t, \mathbf{k})$  not fixed for any  $\mathbf{k}$  lying on the circle radius  $k$  neither is its evolution with respect to time! Hence the main reason for considering higher order terms is to remove the degeneracy of the zeroth order solution.

The next step is to use the above results to obtain equations governing the mean temperature  $\Theta^{(0)}(z, t)$  and the mean horizontal magnetic field  $\mathbf{B}_h^{(0)}(z, t)$ . Since only the zeroth-order terms for the magnetic field are considered the superscript zero is subsequently dropped. The heat conduction equation (2.6 *a*) is considered first. The linear temperature profile is distorted by convection of the fluctuating buoyancy  $\theta^{(0)}$  with the vertical velocity  $w^{(0)}$ . On evaluation of  $\langle w^{(0)}\theta^{(0)} \rangle$  the effect is readily determined from (2.6 *a*) which yields the equation

$$\sigma_\eta \{ \partial \Theta^{(0)} / \partial t + (4\pi k^2 \sigma_\eta \mathcal{F}^{(0)} / \tilde{R}^{(0)}) \sin 2\pi z \} = \partial^2 \Theta^{(0)} / \partial z^2, \quad (3.14)$$

where to lowest order

$$\begin{aligned} \mathcal{F}^{(0)} &= \frac{1}{4} \langle \tilde{\mathbf{u}}_h^{(0)2} + \bar{w}^{(0)2} \rangle, \\ &= \frac{\tilde{R}^{(0)}}{4k^4} \langle \bar{w}^{(0)2} \rangle = \frac{\tilde{R}^{(0)}}{4k^4} \sum_{\mathbf{k}} |\hat{w}^{(0)}(t, \mathbf{k})|^2, \end{aligned} \quad (3.15)$$

defines the mean kinetic energy per unit volume of the fluid. The particular integral of (3.14) which takes account of the convective heat transport is

$$\Theta^{(0)}(z, t) = \hat{\Theta}^{(0)}(t) \sin 2\pi z, \quad (3.16)$$

where

$$\sigma_\eta (d\hat{\Theta}^{(0)}/dt) + 4\pi^2 \hat{\Theta}^{(0)} = -4\pi k^2 \sigma_\eta^2 \mathcal{F}^{(0)} / \tilde{R}^{(0)}. \quad (3.17)$$

An important consequence of (3.17) is that the mean temperature is independent of the individual Fourier modes  $\hat{w}^{(0)}(t, \mathbf{k})$  except in as much as they define the mean kinetic energy of the fluid (see (3.15)). Of course, there are other solutions of (3.14) but since they decay with time they are not investigated. At this point, it is perhaps worth anticipating the result  $\hat{\Theta}^{(0)}(t) < 0$ . It implies that the linear temperature gradient is increased at the boundary. Since this gradient measures the heat flux to and from the fluid, the increased value means that the thermal convection increases the heat transport; a well known property of finite amplitude convection.

In many respects the mean of the magnetic induction equation is similar to (2.6 *a*). The differences, of course, stem from the fact that it is convection of a vector field rather than a scalar field. Thus unlike equation (3.14) which relies on the applied temperature for non-zero values of  $\Theta^{(0)}$ , the magnetic field  $\mathbf{B}_h(z, t)$  is hopefully self-supporting. After all, this is the essence of the dynamo mechanism. Fluctuating magnetic field is induced by stretching the mean magnetic field by the local strain  $\nabla \mathbf{u}^{(0)}$ . The process is described mathematically by equation (2.7 *b*) which is readily solved to give

$$\mathbf{b} = k^{-2} (\mathbf{B}_h \cdot \nabla) \mathbf{u}^{(0)}, \quad (3.18)$$

where

$$\mathbf{u}^{(0)} = \mathbf{i}_z \times \nabla \rho^{(0)} + w^{(0)} \mathbf{i}_z. \quad (3.19)$$

Moreover, convection of the fluctuating magnetic field by the velocity  $\mathbf{u}^{(0)}$  may systematically reinforce the large-scale magnetic field. The effect is determined by the mean quantity

$$\langle \mathbf{u}^{(0)} \times \mathbf{b}^{(0)} \rangle_{\mathbf{h}} = -(\pi/k^6) \mathbf{B}_{\mathbf{h}} \cdot \langle \nabla_{\mathbf{h}} \bar{w}^{(0)} \nabla_{\mathbf{h}} \bar{w}^{(0)} \rangle \sin 2\pi z, \quad (3.20)$$

which appears in (2.7a). Tensors are introduced to keep the notation concise and the suffixes 1 and 2 are used to denote the  $x$  and  $y$  components of a vector respectively. Consequently with the aid of (3.20), equation (2.7a) reduces to

$$\frac{\partial B_i}{\partial t} + 2\pi A \frac{\partial}{\partial z} (\sin 2\pi z M_{ij} B_j) - \frac{\partial^2 B_i}{\partial z^2} = 0, \quad (3.21)$$

where

$$A = \mathcal{F}^{(0)} / \tilde{R}^{(0)}, \quad M_{ij} = \begin{bmatrix} -\alpha_{21} & -\alpha_{22} \\ \alpha_{11} & \alpha_{12} \end{bmatrix}, \quad (3.22a)$$

$$\alpha_{ij} = \frac{\tilde{R}^{(0)}}{2k^6 \mathcal{F}^{(0)}} \langle \nabla_{\mathbf{h}} \bar{w}^{(0)} \nabla_{\mathbf{h}} \bar{w}^{(0)} \rangle_{ij} = \sum_{\mathbf{k}} \frac{k_i k_j}{k^2} q(t, \mathbf{k}), \quad (3.22b)$$

and

$$q(t, \mathbf{k}) = \frac{\tilde{R}^{(0)}}{2k^4 \mathcal{F}^{(0)}} |\hat{w}^{(0)}(t, \mathbf{k})|^2. \quad (3.22c)$$

The normalized quantities  $q(t, \mathbf{k})$  have the property that

$$\alpha_{11} + \alpha_{22} = \sum_{\mathbf{k}} q(t, \mathbf{k}) = 2:$$

the 2 reflects the fact that  $q(t, \mathbf{k}) = q(t, -\mathbf{k})$ . An alternative formulation of (3.21) is obtained with the single equation

$$\begin{aligned} \frac{\partial \phi}{\partial t} + 2\pi i A \alpha \frac{\partial}{\partial z} (\phi \sin 2\pi z) - \frac{\partial^2 \phi}{\partial z^2} \\ = \frac{1}{2} \phi^* \frac{1}{\alpha_{11}} \frac{d\alpha_{11}}{dt} - \frac{1}{2i} (\phi - \phi^*) \left\{ \frac{\alpha_{12}}{\alpha \alpha_{11}} \frac{d\alpha_{11}}{dt} - \frac{1}{\alpha} \frac{d}{dt} (\alpha_{12} + i\alpha) \right\}, \end{aligned} \quad (3.23a)$$

where

$$\alpha_{11}^{\frac{1}{2}} \phi = \alpha_{11} B_1 + (\alpha_{12} + i\alpha) B_2, \quad (3.23b)$$

and

$$\alpha = (\det \alpha_{ij})^{\frac{1}{2}}, \quad (3.23c)$$

takes values between  $-1$  and  $1$ . The second formulation is especially convenient when the flow is independent of time as then the terms on the right vanish. The corresponding reduction of the two variables  $B_1$  and  $B_2$  to the single variable  $\phi$  simplifies the problem as well as reducing the order of the equations. Finally in order to exclude a uniform magnetic field which is outside the scope of dynamo theory (3.23) is solved subject to the initial condition

$$\int_0^1 \phi dz = 0; \quad (3.24)$$

an integral constraint which subsequently holds true for all time.

When  $\alpha_{ij}$  and  $A$  are constants, solutions of (3.23) may be sought in the form

$$\phi = \exp \{s_n \pi^2 t\} \phi_n(z), \quad (3.25)$$

which leads to an eigenvalue problem for  $s_n$ . Evidently there exist steady solutions

$$\phi = \exp \{-i\alpha A \cos 2\pi z\}, \quad (3.26)$$

provided  $\alpha A$  is a zero of the Bessel function  $J_0(\alpha A)$ ; a consequence of (3.24). However, it is clear from (3.26) that no solution having the  $z$  dependence

$$\phi_n = \sum_{m=0}^{\infty} A_{nm} \cos(2m+1)\pi z, \quad (3.27)$$

is steady. In view of the fact that, when  $A = 0$ , the mode with the slowest decay rate is

$$\phi = \exp\{-\pi^2 t\} \cos \pi z,$$

it may be anticipated that the smallest value of  $\alpha A$ , for which there exists a non-decaying mode ( $\text{Re } s \geq 0$ ), is less than the smallest zero of the Bessel function, namely  $\alpha A \doteq 2.4048$ : while the corresponding eigenfunction  $\phi_n$  takes the form (3.27). Indeed numerical calculations (see appendix B) confirm the prediction and it is found that the minimum value of  $\alpha A$  is 1.5974... Since  $|\alpha| \leq 1$ , this implies that the smallest value of  $A$ , for which regeneration is possible, is

$$A_{\text{crit}} = 1.5974\dots, \quad (3.28a)$$

and corresponds to

$$s_0 = i(\omega_0/\pi^2) \doteq -1.3936i. \quad (3.28b)$$

Analysis of dynamo action by a steady velocity field suggests that a key parameter is the magnitude of the product  $\alpha A$ . This is not the case for unsteady velocity fields. Indeed it is possible that periodic dynamos exist when  $\alpha A$  is zero and for values of  $A$  less than  $A_{\text{crit}}$ ! Consider, for example, the case of motion made up of a single roll whose axis rotates with a constant angular velocity  $\omega$ . Such a motion is characterized by  $q(t, \mathbf{k}) = \delta(\theta - \omega t) + \delta(\theta + \pi - \omega t)$ , where  $\mathbf{k} = (k \cos \theta, k \sin \theta)$  and the property that  $\alpha = 0$ . A solution of (3.21) may be sought in the form

$$\mathbf{B}_h(z, t) = B_{\parallel}(z) \mathbf{k}(\omega t) + B_{\perp}(z) \mathbf{i}_z \times \mathbf{k}(\omega t), \quad (3.29)$$

where the components  $B_{\parallel}$  and  $B_{\perp}$  satisfy the equations

$$-\omega B_{\perp} = \partial^2 B_{\parallel} / \partial z^2, \quad (3.30a)$$

$$\omega B_{\parallel} + 4\pi A \frac{\partial}{\partial z} (B_{\parallel} \sin 2\pi z) = \frac{\partial^2 B_{\perp}}{\partial z^2}. \quad (3.30b)$$

The assumed flow is a special case of a more general class of fluid motions considered in §8. Indeed equations (3.30) are obtained from (8.8) when  $X = 1$ ,  $Y = 0$ . Without going into a detailed discussion about the equations, which is left until §8, it is evident that with a sufficiently large value of  $A$  dynamo action can be expected. For small values of  $A$  all the eigenvalues,  $\omega$ , come in complex conjugate pairs and complex  $\omega$  is physically meaningless. When  $A$  equals  $A_{\text{rot}}$  where

$$A_{\text{rot}} \doteq 1.0824, \quad (3.31a)$$

one pair of complex eigenvalues coalesce to give a repeated real eigenvalue which corresponds to the frequency

$$\omega/\pi^2 \doteq -0.9255. \quad (3.31b)$$

When  $A$  is increased still further  $\omega/\pi^2$  splits into two distinct real eigenvalues, i.e. two possible regenerative modes. It is this property which distinguishes the eigenvalue problem (3.30) from (3.23) when  $\mathbf{u}^{(0)}$  is independent of time. In the latter case  $\omega/\pi^2$  is only real when  $A$  takes discrete values; the smallest being  $A_{\text{crit}}$ . The omission of the term  $\phi^*$  in (3.23a) which reduces the order of the equation leads to a degeneracy. Thus, whereas (3.30) determines  $\mathbf{B}_h$  uniquely except for its magnitude, the solution of  $\phi$  in the previous paragraphs is only fixed up to a complex constant.

Consequently the magnitude of the magnetic field has two degrees of freedom; one its magnitude, the other its phase. One final point may be made: though a solution of the type (3.29) is extremely simple, it is special when regarded as a kinematic dynamo. In general it is reasonable to prescribe both  $\omega$  and  $\Lambda$  in which case (3.29) cannot be expected to provide a solution. In spite of these remarks, the model is a prototype of a more general class of motions which arise naturally when the hydromagnetic dynamo is considered.

#### 4. THE FINITE AMPLITUDE SOLUTION

The analysis of the previous section is extended to obtain expressions for second-order quantities such as  $w^{(2)}(\mathbf{x}_h, z, t)$ . It is found that corrections to the buoyancy force due to  $\tilde{R}^{(2)}(k)$  and to the non-uniform gradient of the mean temperature  $\partial\Theta^{(0)}/\partial z$  can be in equilibrium provided the kinetic energy  $\mathcal{F}^{(0)}$  takes the constant value defined by (4.11). At this stage the magnitude of the individual Fourier modes  $\hat{w}^{(0)}(t, \mathbf{k})$  is still undetermined, even though the values of the mean quantities  $\Theta^{(0)}$  and  $\mathcal{F}^{(0)}$  are fixed. Thus the magnitude of the flow adjusts in such a way that the vertical heat transport as measured by the Nusselt number  $1 - \epsilon[\partial\Theta^{(2)}/\partial z]_{z=0} + O(\epsilon^2)$  takes a value determined by  $\tilde{R}^{(2)}(k)$ .

The order  $\epsilon^{\frac{1}{2}}$  terms in (2.4a) are again in geostrophic balance,

$$\mathbf{u}_h^{(1)} = \mathbf{i}_z \times \nabla p^{(1)}.$$

The order  $\epsilon^{\frac{3}{2}}$  terms yield

$$(\eta/\nu) \mathbf{u}_h^{(0)} \cdot \nabla \mathbf{u}^{(0)} + \mathbf{i}_z \times \mathbf{u}_h^{(3)} = -\nabla p^{(3)} + \nabla_h^2 \mathbf{u}^{(1)}, \quad (4.1)$$

which has the solution

$$\mathbf{u}_h^{(3)} = \mathbf{i}_z \times \nabla \xi^{(3)} + \nabla_h[\nabla_h^2 p^{(1)}], \quad (4.2)$$

where

$$\xi^{(3)} = p^{(3)} + (\eta/\nu) (k^2 p^{(0)2} + \frac{1}{4} \nabla_h^2 p^{(0)2}). \quad (4.3)$$

Moreover, since the nonlinear interactions  $\mathbf{u}_h^{(0)} \cdot \nabla w^{(0)}$  and  $\mathbf{u}_h^{(0)} \cdot \nabla \theta^{(0)}$  vanish, the single equation governing the first-order solution is identical to (3.10): no inhomogeneities occur on the right-hand side of the equation. Consequently, the first-order terms can be assumed to be zero without loss of generality.

The analysis continues, duplicating the arguments (3.1) to (3.11) at the  $\epsilon$ -level. First the order  $\epsilon^2$  terms in (2.4a) give

$$\mathbf{u}_h^{(4)} = \mathbf{i}_z \times \nabla_h p^{(4)} - \mathbf{i}_z \times \nabla_h^2 \mathbf{u}_h^{(2)}, \quad (4.4)$$

and substitution into the continuity equation (2.8) leads to

$$\partial w^{(2)}/\partial z = -k^4 p^{(2)}, \quad (4.5)$$

where, as in §3, it is assumed that

$$\nabla_h^2 p^{(2)} = -k^2 p^{(2)}. \quad (4.6)$$

The equations corresponding to (3.8) and (3.9) are

$$-\partial p^{(2)}/\partial z + \sigma_\eta^{-1} \tilde{R}^{(0)} \theta^{(2)} + \nabla_h^2 w^{(2)} = -\sigma_\eta^{-1} \tilde{R}^{(2)} \theta^{(0)} = -(\tilde{R}^{(2)}/k^2) \bar{w}^{(0)} \sin \pi z, \quad (4.7)$$

$$\sigma_\eta w^{(2)} + \nabla_h^2 \theta^{(2)} = \sigma_\eta w^{(0)} (d\Theta^{(0)}/dz) = \pi \sigma_\eta \Theta^{(0)} \bar{w}^{(0)} (\sin 3\pi z - \sin \pi z). \quad (4.8)$$

The equations (4.5) to (4.8) are dominated by the buoyancy forces on the right of (4.7) and (4.8). The only nonlinear interaction results from the convection of the mean horizontal temperature

$\Theta^{(0)}(z, t)$ . Hence no new horizontal wavenumbers are introduced and the fact provides justification for the assumption (4.6). The equations (4.5) to (4.8) have the solution

$$w^{(2)}(\mathbf{x}_h, z, t) = \bar{w}^{(2)}(\mathbf{x}_h, t) \sin \pi z + \frac{k^2}{8\pi^2} \tilde{R}^{(2)} \bar{w}^{(0)}(\mathbf{x}_h, t) \sin 3\pi z, \quad (4.9a)$$

$$p^{(2)}(\mathbf{x}_h, z, t) = -\frac{\pi}{k^4} \bar{w}^{(2)}(\mathbf{x}_h, t) \cos \pi z - \frac{3}{8\pi k^2} \tilde{R}^{(2)} \bar{w}^{(0)}(\mathbf{x}_h, t) \cos 3\pi z, \quad (4.9b)$$

$$\theta^{(2)}(\mathbf{x}_h, z, t) = \frac{\sigma_\eta}{k^2} \left\{ \bar{w}^{(2)}(\mathbf{x}_h, t) - \frac{\tilde{R}^{(2)}}{\tilde{R}^{(0)}} \bar{w}^{(0)}(\mathbf{x}_h, t) \right\} \sin \pi z + \sigma_\eta \frac{k^6 + 9\pi^2 \tilde{R}^{(2)}}{k^6 + \pi^2} \frac{\tilde{R}^{(2)}}{8\pi^2} \bar{w}^{(0)}(\mathbf{x}, t) \sin 3\pi z, \quad (4.9c)$$

provided  $\hat{\Theta}^{(0)}(t)$  takes constant value

$$\hat{\Theta}^{(0)} = -\tilde{R}^{(2)}/\pi \tilde{R}^{(0)}. \quad (4.10)$$

It follows from (3.17) that the kinetic energy is

$$\mathcal{F}^{(0)} = \tilde{R}^{(2)}/k^2 \sigma_\eta^2, \quad (4.11)$$

which is also constant. Since  $\mathcal{F}^{(0)}$  is only positive when  $\tilde{R}^{(2)}$  is greater than zero, it is to be expected that the equilibrium is stable.

Now that  $w^{(2)}$  and  $\theta^{(2)}$  are known, it is possible to determine  $\Theta^{(2)}(z, t)$ . The quantity

$$\langle w^{(0)}\theta^{(2)} + w^{(2)}\theta^{(0)} \rangle$$

is computed and substituted into the mean equation (2.6a). The equation corresponding to (3.14) is then

$$\frac{\partial \Theta^{(2)}}{\partial t} + \frac{k^4 \sigma_\eta^3 \mathcal{F}^{(0)^2}}{\pi \tilde{R}^{(0)^2}} \{2(k^6 + 5\pi^2) \sin 4\pi z - (k^6 + 9\pi^2) \sin 2\pi z\} + \frac{4\pi k^2 \sigma_\eta}{\tilde{R}^{(0)}} \mathcal{F}^{(2)} \sin 2\pi z = \sigma_\eta^{-1} \frac{\partial^2 \Theta^{(2)}}{\partial z^2}, \quad (4.12)$$

where 
$$\mathcal{F}^{(2)} = (\tilde{R}^{(0)}/2k^4) \langle \bar{w}^{(2)} \bar{w}^{(0)} \rangle. \quad (4.13)$$

Again the free decay modes are ignored and the particular integral forced by the convection is

$$\Theta^{(2)} = -\frac{k^4 \mathcal{F}^{(0)^2} \sigma_\eta^4}{8\pi^3 \tilde{R}^{(0)^2}} \{ (k^6 + 5\pi^2) \sin 4\pi z - 2(k^6 + 9\pi^2) \sin 2\pi z \} + \hat{\Theta}^{(2)}(t) \sin 2\pi z, \quad (4.14)$$

where 
$$\frac{d\hat{\Theta}^{(2)}}{dt} + 4\pi^2 \sigma_\eta^{-1} \hat{\Theta}^{(2)} = -\frac{4\pi k^2 \sigma_\eta}{\tilde{R}^{(0)}} \mathcal{F}^{(2)}. \quad (4.15)$$

Whereas in §3,  $\hat{\Theta}^{(0)}(t)$  and  $\mathcal{F}^{(0)}$  were left undetermined, their values are fixed but now it is the corresponding quantities  $\hat{\Theta}^{(2)}(t)$  and  $\mathcal{F}^{(2)}$  which are unknown! The precise form of the velocity as defined by the Fourier modes  $\hat{w}^{(0)}(t, \mathbf{k})$  is still undetermined; however the kinetic energy, as defined by  $\mathcal{F}^{(0)}$ , provides an important constraint on possible motion.

## 5. THE EVOLUTION OF THE VELOCITY FIELD

Since the zeroth-order quantities are still unspecified, the analysis of the equations is continued until equations governing the order  $\epsilon^2$  quantities such as  $w^{(4)}$  are obtained. The equations are not solved but the condition that they have a solution leads to equations governing the evolution of  $\hat{w}^{(0)}(t, \mathbf{k})$ . Thus the prime objective is the determination of equation (5.12). Some of the coefficients appearing in this equation have lengthy algebraic expressions which are derived after



tedious manipulation. However the details of the calculations are unimportant and with the exception of the terms arising from the Lorentz force it is only certain gross features of the coefficients that are needed for the subsequent analysis.

First the equations governing the  $\epsilon^{\frac{5}{2}}$  terms are

$$\frac{\eta}{\nu} \left\{ w^{(0)} \frac{\partial \mathbf{u}_h^{(0)}}{\partial z} + \mathbf{u}_h^{(2)} \cdot \nabla \mathbf{u}_h^{(0)} + \mathbf{u}_h^{(0)} \cdot \nabla \mathbf{u}_h^{(2)} \right\} + \mathbf{i}_z \times \mathbf{u}^{(5)} = -\nabla_h p^{(5)} + \nabla_h^2 \mathbf{u}_h^{(3)}, \quad (5.1a)$$

$$\frac{\eta}{\nu} \left\{ \frac{\partial}{\partial z} (w^{(0)2} - \langle w^{(0)2} \rangle) + \nabla_h \cdot (w^{(0)} \mathbf{u}_h^{(2)} + w^{(2)} \mathbf{u}_h^{(0)}) \right\} = -\frac{\partial p^{(3)}}{\partial z} + \sigma_\eta^{-1} \tilde{R}^{(0)} \theta^{(3)} + \nabla_h^2 w^{(3)}, \quad (5.1b)$$

$$\sigma_\eta \left\{ \frac{\partial}{\partial z} (w^{(0)} \theta^{(0)} - \langle w^{(0)} \theta^{(0)} \rangle) + \nabla_h \cdot (\theta^{(0)} \mathbf{u}_h^{(2)} + \theta^{(2)} \mathbf{u}_h^{(0)}) \right\} - \sigma_\eta w^{(3)} = \nabla_h^2 \theta^{(3)}. \quad (5.1c)$$

After some simplification (5.1a) becomes

$$\mathbf{i}_z \times \left\{ \mathbf{u}^{(5)} + \frac{\eta}{\nu} \frac{1}{2k^4} \nabla_h \left[ \pi^2 w^{(0)2} + \left( \frac{\partial w^{(0)}}{\partial z} \right)^2 \right] \right\} = -\nabla_h \xi^{(5)} + \mathbf{i}_z \times \nabla (\nabla_h^2 \xi^{(3)}), \quad (5.2a)$$

where 
$$\xi^{(5)} = p^{(5)} + \frac{\eta}{\nu} \left\{ 2k^2 p^{(0)} p^{(2)} + \frac{1}{2} \nabla_h^2 (p^{(0)} p^{(2)}) \right\}. \quad (5.2b)$$

The continuity equation is used to eliminate  $\mathbf{u}_h^{(5)}$  and  $\xi^{(5)}$  and leads to the equation

$$\frac{\partial w^{(3)}}{\partial z} + \nabla_h^4 p^{(3)} = -\frac{\eta}{\nu} \frac{\pi^2}{k^8} \left[ \left\{ \frac{1}{4} \nabla_h^6 \bar{w}^{(0)2} + k^2 \nabla_h^4 \bar{w}^{(0)2} \right\} \cos^2 \pi z - \frac{1}{2} k^4 \nabla_h^2 \bar{w}^{(0)2} \right]. \quad (5.3)$$

Similar reductions to (5.1b) and (5.1c) lead to

$$-\frac{\partial p^{(3)}}{\partial z} + \sigma_\eta^{-1} \tilde{R}^{(0)} \theta^{(3)} + \nabla_h^2 w^{(3)} = \frac{\eta}{\nu} \frac{\pi}{k^2} \left\{ \frac{1}{4} \nabla_h^2 \bar{w}^{(0)2} + k^2 (\bar{w}^{(0)2} - \langle \bar{w}^{(0)2} \rangle) \right\} \sin 2\pi z, \quad (5.4)$$

and 
$$\sigma_\eta w^{(3)} + \nabla_h^2 \theta^{(3)} = \sigma_\eta^2 \frac{\pi}{k^4} \left\{ \frac{1}{4} \nabla_h^2 \bar{w}^{(0)2} + k^2 (\bar{w}^{(0)2} - \langle \bar{w}^{(0)2} \rangle) \right\} \sin 2\pi z, \quad (5.5)$$

while further elimination of  $\theta^{(3)}$  and  $p^{(3)}$  from (5.3) to (5.5) yields the single equation

$$\frac{\partial^2 w^{(3)}}{\partial z^2} - \tilde{R}^{(0)} \nabla_h^2 w^{(3)} + \nabla_h^4 w^{(3)} = \frac{\eta}{\nu} \frac{\pi}{4k^8} (k^6 + \pi^2) \nabla_h^2 \left[ \nabla_h^4 \bar{w}^{(0)2} + (4 - \sigma) k^2 \nabla_h^2 \bar{w}^{(0)2} - 4\sigma k^4 \bar{w}^{(0)2} \right] \sin 2\pi z. \quad (5.6)$$

Finally the particular integral of (5.6) is just

$$w^{(3)} = \frac{\eta}{\nu} \frac{\pi}{4} \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \frac{|\mathbf{k}_1 + \mathbf{k}_2|^2}{k^4} V(|\mathbf{k}_1 + \mathbf{k}_2|) \hat{w}^{(0)}(t, \mathbf{k}_1) \hat{w}^{(0)}(t, \mathbf{k}_2) e^{i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{x}} \sin 2\pi z, \quad (5.7)$$

where 
$$V(K) = \frac{-1}{k^2} \frac{(k^6 + \pi^2)(K^2 - 4k^2)(\sigma k^2 + K^2)}{\pi^2(K^2 - 4k^2) + K^2 k^2(k^4 - K^4)}. \quad (5.8)$$

The other variables  $p^{(3)}$ ,  $\xi^{(3)}$  and  $\theta^{(3)}$  are readily obtained from (5.3) to (5.8).

The investigation proceeds to the  $\epsilon^3$  terms. At this level of approximation all effects previously neglected are important. The most notable are the terms involving the time derivative and the Lorentz force. The equations corresponding to (5.1) are

$$\mathcal{F}_1 + \mathbf{i}_z \times \mathbf{u}_h^{(6)} = -\nabla_h p^{(6)} + \nabla_h^2 \mathbf{u}_h^{(4)}, \quad (5.9a)$$

$$\mathcal{F}_2 = -\partial p^{(4)} / \partial z + \sigma_\eta^{-1} \tilde{R}^{(0)} \theta^{(4)} + \nabla_h^2 w^{(4)}, \quad (5.9b)$$

$$\mathcal{F}_3 - \sigma_\eta w^{(4)} = \nabla_h^2 \theta^{(4)}, \quad (5.9c)$$

where

$$\mathcal{F}_1 = \frac{\eta}{\nu} \left\{ \frac{\partial \mathbf{u}_h^{(0)}}{\partial t} + \mathbf{u}_h^{(3)} \cdot \nabla \mathbf{u}_h^{(0)} + \mathbf{u}_h^{(0)} \cdot \nabla \mathbf{u}_h^{(3)} \right\} - \frac{M^2}{k^2} (\mathbf{B}_h \cdot \nabla)^2 \mathbf{u}_h^{(0)} - \frac{\partial^2 \mathbf{u}_h^{(0)}}{\partial z^2}, \quad (5.10a)$$

$$\mathcal{F}_2 = \frac{\eta}{\nu} \left\{ \frac{\partial w^{(0)}}{\partial t} + \mathbf{u}_h^{(3)} \cdot \nabla w^{(0)} + \mathbf{u}_h^{(0)} \cdot \nabla w^{(3)} \right\} - \sigma_\eta^{-1} \tilde{R}^{(2)} \theta^{(2)} - \sigma_\eta^{-1} \tilde{R}^{(4)} \theta^{(0)} - \frac{M^2}{k^2} (\mathbf{B}_h \cdot \nabla)^2 w^{(0)} - \frac{\partial^2 w^{(0)}}{\partial z^2}, \quad (5.10b)$$

$$\mathcal{F}_3 = \sigma_\eta \left\{ \frac{\partial \theta^{(0)}}{\partial t} + \mathbf{u}_h^{(3)} \cdot \nabla \theta^{(0)} + \mathbf{u}_h^{(0)} \cdot \nabla \theta^{(3)} + w^{(0)} \frac{\partial \theta^{(2)}}{\partial z} + w^{(2)} \frac{\partial \theta^{(0)}}{\partial z} \right\} - \frac{\partial^2 \theta^{(0)}}{\partial z^2}, \quad (5.10c)$$

which can be solved provided

$$2 \left\langle \int_0^1 \exp\{-i\mathbf{k} \cdot \mathbf{x}\} \sin \pi z \left[ \nabla_h \cdot \left( \mathbf{i}_z \times \frac{\partial \mathcal{F}_1}{\partial z} \right) + \sigma_\eta^{-1} \tilde{R}^{(0)} \nabla_h^2 \mathcal{F}_3 - \nabla_h^4 \mathcal{F}_2 - \pi^2 k^4 w^{(0)} \right] dz \right\rangle = 0, \quad (5.11)$$

for all  $|\mathbf{k}| = k$ . Direct evaluation of (5.11) leads to equations governing  $\hat{w}^{(0)}(t, \mathbf{k})$ . The nonlinear terms involving  $w^{(3)}$  lead to cubic expressions. However, most of these terms vanish as the sum  $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4$ , where  $|k_i| = k$  ( $i = 1, \dots, 4$ ), is zero only if the corresponding vector diagram forms a parallelogram. The resulting equation is

$$d\hat{w}^{(0)}/dt + \sum_{\mathbf{k}'} A(\mathbf{k}', \mathbf{k}) |\hat{w}^{(0)}(t, \mathbf{k}')|^2 \hat{w}^{(0)} - \frac{1}{2} m(t, \mathbf{k}) w^{(0)} + \frac{1}{2} r(k) \hat{w}^{(0)} - \frac{1}{2} \beta(k) \hat{\Theta}^{(2)}(t) \hat{w}^{(0)} = 0, \quad (5.12)$$

where

$$A(\mathbf{k}', \mathbf{k}) = (\eta/\nu)^2 (\mathbf{k} \times \mathbf{k}') \cdot \mathbf{i}_z \Gamma(|\mathbf{k} + \mathbf{k}'|, k, \sigma), \quad (5.13a)$$

$$m(t, \mathbf{k}) = k^{-2} k_i m_{ij} k_j, \quad (5.13b)$$

$$m_{ij} = 4 \frac{\nu}{\eta} M^2 \frac{\int_0^1 (\pi^2 \cos^2 \pi z - k^6 \sin^2 \pi z) B_i B_j dz}{\{\pi^2(\sigma - 1) + k^6(\sigma + 1)\}}, \quad (5.13c)$$

$$r(k) = 2 \frac{\nu}{\eta} \frac{\{3\pi^2 k^6 - k^4 \tilde{R}^{(4)}(k) + (k^6 + 25\pi^2) \sigma_\eta^4 k^8 \mathcal{F}^{(0)2} / (8\pi^2 \tilde{R}^{(0)})\}}{\pi^2(\sigma - 1) + k^6(\sigma + 1)}, \quad (5.13d)$$

$$\beta(k) = 2 \frac{\nu}{\eta} \frac{\tilde{R}^{(0)} k^4}{\pi^2(\sigma - 1) + k^6(\sigma + 1)}. \quad (5.13e)$$

and  $\Gamma(K, k, \sigma)$  is a lengthy algebraic expression involving  $K^2$ ,  $k^2$  and  $\sigma$ .

At first sight the presence of the  $\hat{\Theta}^{(2)}(t)$  term in (5.12) is unexpected. It may be recalled, though, that the kinetic energy is already fixed so that there is an additional relation between the  $\hat{w}^{(0)}(t, \mathbf{k})$ . Hence it is only possible to solve (5.12) if

$$\beta \hat{\Theta}^{(2)}(t) = \frac{1}{2} \sum_{\mathbf{k}} [r(k) - m(t, \mathbf{k})] q(t, \mathbf{k}); \quad (5.14)$$

an identity readily obtained by multiplication of (5.12) with  $\hat{w}^{(0)*}(t, \mathbf{k})$  and summation over  $\mathbf{k}$ . (Note that the time derivative and nonlinear terms vanish because  $\sum_{\mathbf{k}} |\hat{w}^{(0)}(t, \mathbf{k})|^2 = \text{constant}$  and  $A(\mathbf{k}', \mathbf{k}) = -A(\mathbf{k}, \mathbf{k}')$  respectively.)

Equations (3.21), (5.12) and (5.14) govern the evolution of the hydromagnetic dynamo. The equations derived in this section governing the motion are attractive because all dynamical effects are included. However, some care should be taken in the interpretation of (5.12) on account of the term involving  $\hat{\Theta}^{(2)}(t)$ . For example, though magnetic fields are known to relax the constraint of rotation this does not have the effect here of permitting the velocities to increase. Instead buoyancy forces are invoked via the order  $\epsilon$  perturbation  $\Theta^{(2)}(z, t)$  which inhibit any such growth. The rôle of (5.12), is therefore, subtle as it governs the evolution of the velocity

field subject to the constraint that it should lie on a constant energy surface. Of course, the exact values of  $\hat{w}^{(0)}(t, \mathbf{k})$  are of vital importance to the dynamo process. Consequently the role of (5.12) is crucial in the study of the hydromagnetic dynamo as will be seen later in §§7 and 8.

## 6. DYNAMIC STABILITY

The analysis up to this point has pivoted upon the construction of a quasi-steady solution which evolves on the slow diffusion time scale; the filtering approximation. Thus the stability of the flow to order  $\epsilon^{\frac{1}{2}}$  perturbations which vary on the fast time scale  $\epsilon^{\frac{1}{2}}$  is now investigated to justify the procedure.

Most of the results of the previous sections need only small modifications to cope with the new situation. In fact it is only terms involving time derivatives that need reordering. In anticipation of the result  $\hat{w}^{(0)}(t, \mathbf{k})$  is replaced by

$$\hat{w}^{(0)}(t, \mathbf{k}) + \epsilon^{\frac{1}{2}}[\hat{w}_1^{(1)}(t, \mathbf{k}) + \epsilon^{\frac{1}{2}}\hat{w}_1^{(2)}(t, \mathbf{k})] \exp\{i\omega t/\epsilon^{\frac{1}{2}}\} + \epsilon^{\frac{1}{2}}[\hat{w}_{-1}^{(1)}(t, \mathbf{k}) + \epsilon^{\frac{1}{2}}\hat{w}_{-1}^{(2)}(t, \mathbf{k})] \exp\{-i\omega t/\epsilon^{\frac{1}{2}}\} + \epsilon[\hat{w}_0^{(2)}(t, \mathbf{k}) + \hat{w}_2^{(2)}(t, \mathbf{k}) \exp\{2i\omega t/\epsilon^{\frac{1}{2}}\} + \hat{w}_{-2}^{(2)}(t, \mathbf{k}) \exp\{-2i\omega t/\epsilon^{\frac{1}{2}}\}] + \dots, \quad (6.1)$$

while  $\hat{\Theta}^{(2)}(t)$  is replaced by

$$\hat{\Theta}^{(2)}(t) + \text{Re}[(\hat{\Theta}_1^{(2)}(t) + \epsilon^{\frac{1}{2}}\hat{\Theta}_1^{(3)}(t)) \exp\{i\omega t/\epsilon^{\frac{1}{2}}\}] + \epsilon^{\frac{1}{2}} \text{Re}[\hat{\Theta}_0^{(3)}(t) + \hat{\Theta}_2^{(3)}(t) \exp\{2i\omega t/\epsilon^{\frac{1}{2}}\}] + \dots \quad (6.2)$$

The motivation for the expansion of the vertical velocity originates from the necessary violation of the constraint (4.11) on the kinetic energy at the  $\epsilon^{\frac{1}{2}}$  level. Since (3.14) to (3.17) are valid to order  $\epsilon^{\frac{1}{2}}$  the equation

$$\frac{1}{2}i\omega\hat{\Theta}_1^{(2)} = -\sigma_\eta \pi k^{-2} \sum_k (\hat{w}^{(0)*} \hat{w}_1^{(1)} + \hat{w}^{(0)} \hat{w}_{-1}^{(1)*}), \quad (6.3)$$

is readily obtained. Moreover, the rapidly oscillating mean temperature does not effect  $\hat{w}^{(1)}(t, \mathbf{k})$  until (5.12) is considered. Here the order 1 terms fluctuating on the short time scale are equated and yield the equation

$$i\omega\hat{w}_1^{(1)} - \frac{1}{4}\beta\hat{w}^{(0)}\hat{\Theta}_1^{(2)} = 0. \quad (6.4)$$

It follows that

$$\hat{w}_1^{(1)} = -i\frac{\beta}{4\omega}\hat{w}^{(0)}\hat{\Theta}_1^{(2)}, \quad (6.5a)$$

$$\hat{w}_{-1}^{(1)} = i\frac{\beta}{4\omega}\hat{w}^{(0)}\hat{\Theta}_1^{(2)*}, \quad (6.5b)$$

where the frequency of the oscillation is given by

$$\omega^2 = 4\pi\sigma_\eta k^2 \beta \mathcal{F}^{(0)}/\hat{R}^{(0)}. \quad (6.6)$$

The above analysis demonstrates the existence of rapid oscillations which may be superimposed on the quasi-steady solutions and can persist over the diffusion time scale. The vital question, therefore, is whether these oscillations grow or decay. To answer this question the analysis must be extended to the next order. Since it is the elimination of the secular terms which determines an equation for  $\hat{\Theta}^{(2)}(t)$ , only the first harmonics are considered. For these particular terms, equations (3.14) and (5.12) are still valid so the equations corresponding to (6.3) and (6.4) are just

$$\left[\frac{1}{2}i\omega\hat{\Theta}_1^{(3)} + \sigma_\eta \pi k^{-2} \sum_k (\hat{w}^{(0)*} \hat{w}_1^{(2)} + \hat{w}^{(0)} \hat{w}_{-1}^{(2)*})\right] + \frac{1}{2}\frac{d\hat{\Theta}_1^{(2)}}{dt} + 2\pi^2 \sigma_\eta^{-1} \hat{\Theta}_1^{(2)} = 0, \quad (6.7)$$

$$[i\omega\tilde{w}_1^{(2)} - \frac{1}{4}\beta\hat{w}^{(0)}\tilde{\Theta}_1^{(3)}] + \frac{d\tilde{w}_1^{(1)}}{dt} - \frac{i\beta}{4\omega}\left\{3\sum_{\mathbf{k}'} A(\mathbf{k}', \mathbf{k})|\hat{w}^{(0)}(t, \mathbf{k}')|^2 - \frac{1}{2}m(t, \mathbf{k}) + \frac{1}{2}r(k) - \frac{1}{2}\beta\hat{\Theta}^{(2)}(t)\right\}\hat{w}^{(0)}\tilde{\Theta}_1^{(2)} = 0, \quad (6.8)$$

where the substitutions (6.5) for  $\tilde{w}_{\pm 1}^{(1)}$  have already been made in some terms of (6.8). Further simplifications of (6.8) follow from the use of (5.12) which leads to the equation

$$[i\omega\tilde{w}_1^{(2)} - \frac{1}{4}\beta\hat{w}^{(0)}\tilde{\Theta}_1^{(3)}] - \frac{i\beta}{4\omega}\hat{w}^{(0)}\left\{\frac{d\tilde{\Theta}_1^{(2)}}{dt} + 2\sum_{\mathbf{k}'} A(\mathbf{k}', \mathbf{k})|\hat{w}^{(0)}(t, \mathbf{k}')|^2\tilde{\Theta}^{(2)}\right\} = 0. \quad (6.9)$$

Finally  $\tilde{w}_1^{(2)}$  and  $\tilde{\Theta}^{(3)}$  are eliminated from (6.7) and (6.9) giving

$$\frac{d\tilde{\Theta}_1^{(2)}}{dt} + 2\pi^2\sigma_\eta^{-1}\tilde{\Theta}_1^{(2)} = 0, \quad (6.10)$$

where terms involving  $A(\mathbf{k}', \mathbf{k})$  vanish because of the skew symmetry. Since  $\tilde{\Theta}_1^{(2)}$  decays on the diffusion time scale it may be concluded that the model is stable to order  $\epsilon^{\frac{1}{2}}$  perturbations.

## 7. DISCRETE MODAL ANALYSIS

The final two sections of the paper are devoted to a class of solutions of the hydromagnetic dynamo equations derived by asymptotic analysis in the previous sections. There are essentially two sets of equations. First, the evolution of the mean magnetic field  $\mathbf{B}_n^{(0)}(z, t)$  is governed by (3.21). Secondly, solution of this equation requires knowledge of  $q(t, \mathbf{k})$  which is governed by

$$\begin{aligned} \frac{dq}{dt}(t, \mathbf{k}) + 2\sum_{\mathbf{k}'} A(\mathbf{k}', \mathbf{k})q(t, \mathbf{k}')q(t, \mathbf{k}) - [m(t, \mathbf{k}) - \frac{1}{2}\sum_{\mathbf{k}'} m(t, \mathbf{k}')q(t, \mathbf{k}')]q(t, \mathbf{k}) \\ + [r(k) - \frac{1}{2}\sum_{\mathbf{k}'} r(k')q(t, \mathbf{k}')]q(t, \mathbf{k}) = 0, \end{aligned} \quad (7.1)$$

an equation determined from (5.12) to (5.14). Though the mathematical problem posed by these equations is tractable, their nonlinear character and the remaining  $z$ -dependence provide severe obstacles to completely analytic solutions. It may be recalled that, since the layer is unbounded in lateral extent,  $q(t, \mathbf{k})$  is defined for any  $\mathbf{k}$  lying on the circle radius  $k$ . If any  $q(t, \mathbf{k})$  is initially zero however, it remains zero and hence attention may be restricted to a discrete set of modes defined by a finite set of horizontal wavevectors  $\mathbf{k}_n (n = 1, \dots, N)$ . In this case the velocity is defined by

$$\tilde{\mathbf{u}}^{(0)}(\mathbf{x}_n, t) = 2\text{Re} \sum_{n=1}^N \hat{\mathbf{u}}^{(0)}(t, \mathbf{k}_n) \exp\{i\mathbf{k}_n \cdot \mathbf{x}\}. \quad (7.2)$$

When  $N = 1$ , the motion is defined by a single roll whose axis is the fixed direction  $\mathbf{k}_1$  and has constant magnitude  $|\hat{w}(t, \mathbf{k}_1)| = 1$ . Since the dynamical equation (7.1) is redundant, the problem reduces to the kinematic dynamo problem (3.21) for constant  $M_{ij}$ . Unfortunately, though this is the simplest possible case, the dynamo must fail, as  $\alpha = 0$ . Quite simply, the motion lacks the asymmetries necessary for dynamo action. Evidently the motion necessary for a workable dynamo must be represented by two or more such rolls (i.e.  $N \geq 2$ ). If the Rayleigh number (or in particular  $\hat{R}^{(2)}$  and hence  $\mathcal{S}^{(0)}$  and  $A$ ) is sufficiently large and the asymmetric nature of the motion is maintained there is little doubt that the dynamo will succeed. The latter criterion highlights the main difficulty, since in certain problems concerning the stability of finite amplitude Bénard convection it is known that the single roll ( $N = 1$ ) is the only stable case. Therefore, it might be

expected that, if initially a finite number of  $q(t, \mathbf{k}_n)$  are non-zero, all the  $q(t, \mathbf{k}_n)$  except one approach zero, as  $t \rightarrow \infty$ , in which case the dynamo fails. In this section investigation of two models based on two and three rolls show that this need not be the case and demonstrate the existence of stable hydromagnetic dynamos.

In the first model, the motion is the superposition of two perpendicular rolls. Here  $N = 2$  and

$$\mathbf{k}_1 = (k, 0), \quad \mathbf{k}_2 = (0, k). \quad (7.3)$$

When  $\hat{w}^{(0)}(t, \mathbf{k}_1) = \hat{w}^{(0)}(t, \mathbf{k}_2)$ , the resulting motion is confined to square cells. In the second model, the motion is the superposition of three rolls whose axes intersect at an angle of  $\frac{1}{3}\pi$ . Here  $N = 3$  and

$$\mathbf{k}_1 = (k, 0), \quad \mathbf{k}_2 = (-\frac{1}{2}k, \sqrt{3}/2k), \quad \mathbf{k}_3 = (-\frac{1}{2}k, -\sqrt{3}/2k). \quad (7.4)$$

When all three  $\hat{w}^{(0)}(t, \mathbf{k}_n)$  are equal, the resulting motion is again special and confined to hexagonal cells. The first case is treated both analytically and numerically, while only numerical results are presented for the second case. It is the latter case, however, which provides the key to an analytic solution of a continuous modal distribution described in §8.

(a) *Analytic results,  $N = 2$*

The case  $N = 2$  is examined. The substitutions

$$q(t, \mathbf{k}_1) = \frac{1}{2} + v, \quad q(t, \mathbf{k}_2) = \frac{1}{2} - v, \quad (7.5)$$

reduce (7.1) to the more symmetric form

$$\frac{1}{\pi^2} \frac{dv}{dt} + \{\gamma + 4\bar{M}^2 \mathcal{L}(B_2^2 - B_1^2)\} (\frac{1}{4} - v^2) = 0, \quad (7.6a)$$

where

$$\pi^2 \gamma = r(k_1) - r(k_2), \quad (7.6b)$$

$$\bar{M}^2 = \frac{\nu}{\eta} M^2 / \{\pi^2(\sigma - 1) + k^6(\sigma + 1)\}, \quad (7.6c)$$

$$\mathcal{L}(f) = \int_0^1 \left( \cos^2 \pi z - \frac{k^6}{\pi^2} \sin^2 \pi z \right) f(z) dz. \quad (7.6d)$$

The vanishing of all the terms involving  $A(\mathbf{k}', \mathbf{k})$  is exceptional and results from the equality  $A(\mathbf{k}_1, \mathbf{k}_2) = -A(-\mathbf{k}_1, \mathbf{k}_2)$ , peculiar to the case of two perpendicular rolls. Except for the added complication of the  $z$ -dependence, the set of equations (3.21) and (7.6) are reminiscent of the equations governing the Rikitake two-disk dynamo (Rikitake 1958; Cook & Roberts 1970). Solutions of these new equations, however, do not appear to exhibit the feature of sudden reversal characteristic of the two disk dynamo equations.

The case of no magnetic field is readily treated and yields a solution

$$v = -\frac{1}{2} \tanh\left(\frac{1}{2}\pi^2\gamma t\right), \quad (7.7)$$

for  $\gamma \neq 0$ . Since non-zero  $\gamma$  corresponds to two values of  $\mathbf{k}$  differing by order  $\epsilon$  it is clear that stability is ensured only if the magnitude of the two wave numbers are identical. The numerical results of the next section show that when  $\gamma \neq 0$  the presence of magnetic field is able to prevent the ultimate decay of one mode, at the same time providing a periodic dynamo (see figure 3). This example is evidently important as it demonstrates the stabilizing influence of the magnetic

field to an otherwise unstable configuration. Some analytical progress can be made towards understanding the physical mechanisms involved by supposing that

$$M^2 \ll 1, \quad (7.8)$$

and that  $\gamma = 0$ . Though the assumption  $\gamma = 0$  eliminates the source of the instability in the non-magnetic case, it isolates the influence of the Lorentz force when a small magnetic field is present. For steady flow, the square cell provides the most favourable motion for dynamo action. Consequently a perturbation scheme is developed in which

$$A = A_{\text{crit}} + \bar{M}^2 A^{(2)}, \quad (7.9a)$$

$$\text{and} \quad v = \bar{M}^2 v^{(2)} + O(\bar{M}^4). \quad (7.9b)$$

It is convenient to introduce the complex function

$$\phi = (1 + 2v)^{\frac{1}{2}} B_1 + i(1 - 2v)^{\frac{1}{2}} B_2, \quad (7.10)$$

defined by (3.23b), and to consider (3.23a), namely

$$\frac{\partial \phi}{\partial t} + \frac{2v\phi - (\frac{1}{2} + v)\phi^*}{1 - 4v^2} \frac{dv}{dt} + 2\pi i A (1 - 4v^2)^{\frac{1}{2}} \frac{\partial}{\partial z} (\phi \sin 2\pi z) - \frac{\partial^2 \phi}{\partial z^2} = 0, \quad (7.11)$$

in conjunction with

$$\frac{1}{\pi^2} \frac{dv}{dt} + \bar{M}^2 \{ 2v \mathcal{L}(|\phi|^2) - \frac{1}{2} \mathcal{L}(\phi^2 + \phi^{*2}) \} = 0, \quad (7.12)$$

obtained from (7.6) by a change of variables. Evidently the solution of (7.11) takes the form

$$\phi(z, t) = a(\tau) \Phi(z) e^{i\omega_0 t} + O(\bar{M}^2) \quad (\tau = \bar{M}^2 t), \quad (7.13)$$

where  $\Phi(z)$  is the eigenfunction  $\Phi_0(z)$  corresponding to the eigenvalue  $i\omega_0/\pi^2 = s_0$  (see (3.28) and appendix B), when  $A = A_{\text{crit}}$ . Moreover, it transpires that the amplitude  $a(\tau)$ , which is undetermined at this stage, is modulated over a long time scale. The small fluctuation  $\bar{M}^2 v^{(2)}$ , measuring small changes in the individual amplitudes of the two rolls is just

$$v^{(2)} = \text{Re} \left\{ \frac{\pi^2 a^2 \mathcal{L}(\Phi^2)}{2i\omega_0} e^{2i\omega_0 t} \right\}, \quad (7.14)$$

which is readily determined from the order  $\bar{M}^2$  terms in (7.12). Before the order  $\bar{M}^2$  correction to  $\phi$  can be obtained from (7.11), the condition that the secular terms should be eliminated provides an equation governing  $a(\tau)$ . This is achieved most simply by multiplying (7.11) by  $\Phi^A(z)$ , the adjoint eigenfunction defined in appendix B and taking the  $z$ -average. The order  $\bar{M}^2$  terms of the resulting expression yield the equation

$$\frac{1}{\pi^2} \frac{da}{d\tau} - \frac{1}{2} A |a|^2 a + \frac{1}{2} C a = 0, \quad (7.15)$$

where

$$A = \frac{1}{2} \frac{(\Phi^A, \Phi^*)}{(\Phi^A, \Phi)} \mathcal{L}(\Phi^2), \quad (7.16a)$$

$$C = -\frac{2}{\pi^2} \frac{A^{(2)}}{A_{\text{crit}}} \left\{ \frac{(\partial \Phi^A / \partial z, \partial \Phi / \partial z)}{(\Phi^A, \Phi)} + i\omega_0 \right\}, \quad (7.16b)$$

and

$$(\Phi^A, \Phi) = \int_0^1 \Phi^A(z) \Phi(z) dz. \quad (7.16c)$$

Numerical values of the complex numbers  $A$  and  $C$  are given in appendix B. In particular, when  $k = k_c$  (the case of principal interest), the real part of  $A$  is positive and the real part of  $C$  is opposite to the sign of  $A^{(2)}$ .

An equation governing the amplitude  $|a(\tau)|$  is derived from (7.15). In terms of the new variable  $X = |a|^2$  the equation is

$$\frac{1}{\pi^2} \frac{dX}{d\tau} = \operatorname{Re} A (X - X_0) X, \quad (7.17)$$

where  $X_0 = \operatorname{Re} C / \operatorname{Re} A$ . There is an equilibrium point at  $X = 0$ , and (provided  $A^{(2)} < 0$ ) an unstable equilibrium at  $X = X_0$ . In the latter case the phase of  $a(\tau)$  may vary with time and can be regarded as providing an order  $\bar{M}^2$  correction to the frequency  $\omega$ . If  $0 < X < X_0$ , then  $X$  approaches zero, as  $t \rightarrow \infty$  in which case the dynamo fails, while if  $0 < X_0 < X$ , then  $X$  tends to  $\infty$ , as  $t \rightarrow \infty$ , in which case all that can be concluded is that the field initially grows. Eventually the magnetic field must go outside the range for which the present asymptotic treatment is valid.

The various terms in (7.15) deserve interpretation. Clearly the linear terms are readily understood on linear theory. In particular, if the magnetic Reynolds number is increased above its critical value the field grows. The nonlinear term involving  $A$  may be traced to the effect of a time-dependent velocity in the magnetic induction equation. Within the framework of the perturbation analysis it is possible for the magnetic field to grow even when  $A < A_{\text{crit}}$ . Therefore it is to be expected that the minimum value of  $A$  for which hydromagnetic dynamo action can be sustained indefinitely is less than  $A_{\text{crit}}$  and corresponds to a fully time-dependent motion. The conclusion is hardly surprising when the results of the kinematic dynamo problem for a time-dependent motion in §3 are recalled. Moreover, one might ask if  $A_{\text{rot}}$  (see (3.31)) provides the minimum value of  $A$  for which hydromagnetic dynamos are possible and if it is attained? A partial answer to the question is provided in §8.

What is the ultimate fate of the dynamo described initially by (7.17) when  $0 < X_0 < X$ ? First, the magnetic field  $\mathbf{b}$  may become order 1 in which case a study of (7.6) and (3.21) is adequate. Secondly,  $\mathbf{b}$  may continue to increase beyond this value in which case the asymptotic analysis based on  $M$  order 1 is inadequate i.e. the analysis of the weak field limit becomes inappropriate. Thirdly, the magnetic field may ultimately decay to zero; a possibility suggested by the case of a single roll. The second possibility is outside the scope of the present paper, but is at present being investigated by Childress (1974). The first possibility is considered in detail in §§7*b, c* by numerical integration of the governing equations and analytically in §8.

#### (b) Numerical results, $N = 2$

In §7*a* it was natural to base a perturbation procedure on equations (7.11) and (7.12). However, for the purpose of numerical calculations, it is easier to consider (7.6) in conjunction with (3.21). In order to isolate the most unstable modes it is supposed that the magnetic field can be expressed in the form

$$\mathbf{B}_h(z, t) = \sum_{m=0}^{\infty} \mathbf{B}_h^{(m)}(t) \cos(2m+1)\pi z. \quad (7.18)$$

The representation filters on the even modes which presumably decay anyway for the relatively small values of  $A$  considered. Though a restricted class of solutions is investigated, it is reasonable to suppose, on the basis of the kinematic analysis, that the most readily excited mode is even.

The dynamo equation may be reduced to an infinite set of equations governing  $\mathbf{B}_h^{(m)}(t)$ . These are

$$\frac{1}{\pi^2} \frac{d\mathbf{B}_i^{(m)}}{dt} + \Lambda M_{ij} \sum_{n=0}^{\infty} K^{(m,n)} \mathbf{B}_j^{(n)} + (2m+1)^2 \mathbf{B}_i^{(m)} = 0, \quad (7.19a)$$

where

$$M_{ij} = \begin{bmatrix} 0 & -1 + 2v \\ 1 + 2v & 0 \end{bmatrix},$$

$$-K^{(m, m+1)} = K^{(m, m-1)} = 2m + 1, \quad (7.19b)$$

$$K^{(0, 0)} = 1, \quad K^{(m, n)} = 0 \quad \text{otherwise,}$$

while the quantities  $\mathcal{L}(B_1^2)$  and  $\mathcal{L}(B_2^2)$  in (7.6) are defined by

$$4\mathcal{L}(B_i B_j) = (1 - k^6/\pi^2) \left( \sum_{m=0}^{\infty} B_i^{(m)} B_j^{(m)} \right) + \frac{1}{2}(1 + k^6/\pi^2) \left\{ B_i^{(0)} B_j^{(0)} + \sum_{m=0}^{\infty} (B_i^{(m)} B_j^{(m+1)} + B_j^{(m)} B_i^{(m+1)}) \right\}. \quad (7.20)$$

In all subsequent calculations the free parameter  $k$  is set equal to  $k_c$ , thus restricting attention to the most unstable modes.

The numerical calculations were performed by taking only the first ten harmonics. Specifically, the approximation involves setting  $B_h^{(m)} = 0$  when  $m \geq 10$ . The resulting twenty-one equations were integrated numerically by the Runge-Kutta method with  $B_h^{(m)}(0)$  ( $0 \leq m < 10$ ) and  $v(0)$  given.

A partial check of the numerical scheme is provided by reproducing the results of the eigenvalue problem already considered in §3. The initial value problem

$$\sum_{m=0}^{\infty} (B_1^{(m)}(0) + iB_2^{(m)}(0)) \cos(2m+1)\pi z = \Phi(z), \quad (7.21a)$$

$$v(0) = 0, \quad (7.21b)$$

where the coefficients  $B_1^{(m)}(0)$  and  $B_2^{(m)}(0)$  are listed in the first two columns of table 1 (see appendix B), is considered for the case  $\Lambda = \Lambda_{\text{crit}}$ ,  $\gamma = \bar{M}^2 = 0$ . The values of  $B_1^{(m)}(t)$  and  $B_2^{(m)}(t)$  after a time  $t_0 = 2\pi/\omega_0$  (the period of an oscillation) are compared with their initial values. With a time step  $\Delta t = t_0/800$ , agreement to seven decimal places may be obtained. It is reasonable to suppose that, with the addition of just the single equation (7.6), integration of the full nonlinear system is of comparable accuracy.

Results for  $\gamma = 0$  indicate that hydromagnetic dynamos can be sustained with  $\Lambda < \Lambda_{\text{crit}}$ . The results of a calculation with  $\Lambda = 1.548$ ,  $\bar{M}^2 = 1.9$  and  $\Delta t = t_0/1600$  are shown in figure 2. The same initial values of  $B_h^{(m)}(0)$  are taken as in the kinematic problem. Indeed, these initial conditions are adopted for all initial value problems considered in this section. By a time  $15t_0$  the solution appears to be rapidly approaching a limit cycle, in which the period is approximately  $0.9t_0$  and the average magnetic energy

$$E(t) = \frac{1}{2} \bar{M}^2 \int_0^1 B_h^2 dz$$

fluctuates slightly about the value 0.7089 attained when  $v = 0$ . It is interesting to note that the magnetic energy is increased when  $v$  takes its largest values close to  $v = \pm \frac{1}{4}$ , where  $\det M_{ij}$  is small. This is contrary to the results of the kinematic dynamo with a steady velocity field for which  $(1 - 4v^2)^{\frac{1}{2}} \Lambda$  is the crucial parameter, but is consistent with remarks made about unsteady velocity fields at the end of §3.

The physical process is readily understood. First,  $v$  increases (decreases) when the magnetic field  $B_1^{(m)}$ , as measured by  $\mathcal{L}(B_1^2)$ , is greater (less) than  $B_2^{(m)}$ . It is also evident from (7.19) that creation of the field  $B_2^{(m)}$  is most favourable when  $v$  is large and positive and  $B_1^{(m)}$  is also in some sense large. Now, as  $v$  increases to its maximum value, where  $\mathcal{L}(B_1^2) = \mathcal{L}(B_2^2)$ ,  $B_1^{(m)}$  is decaying



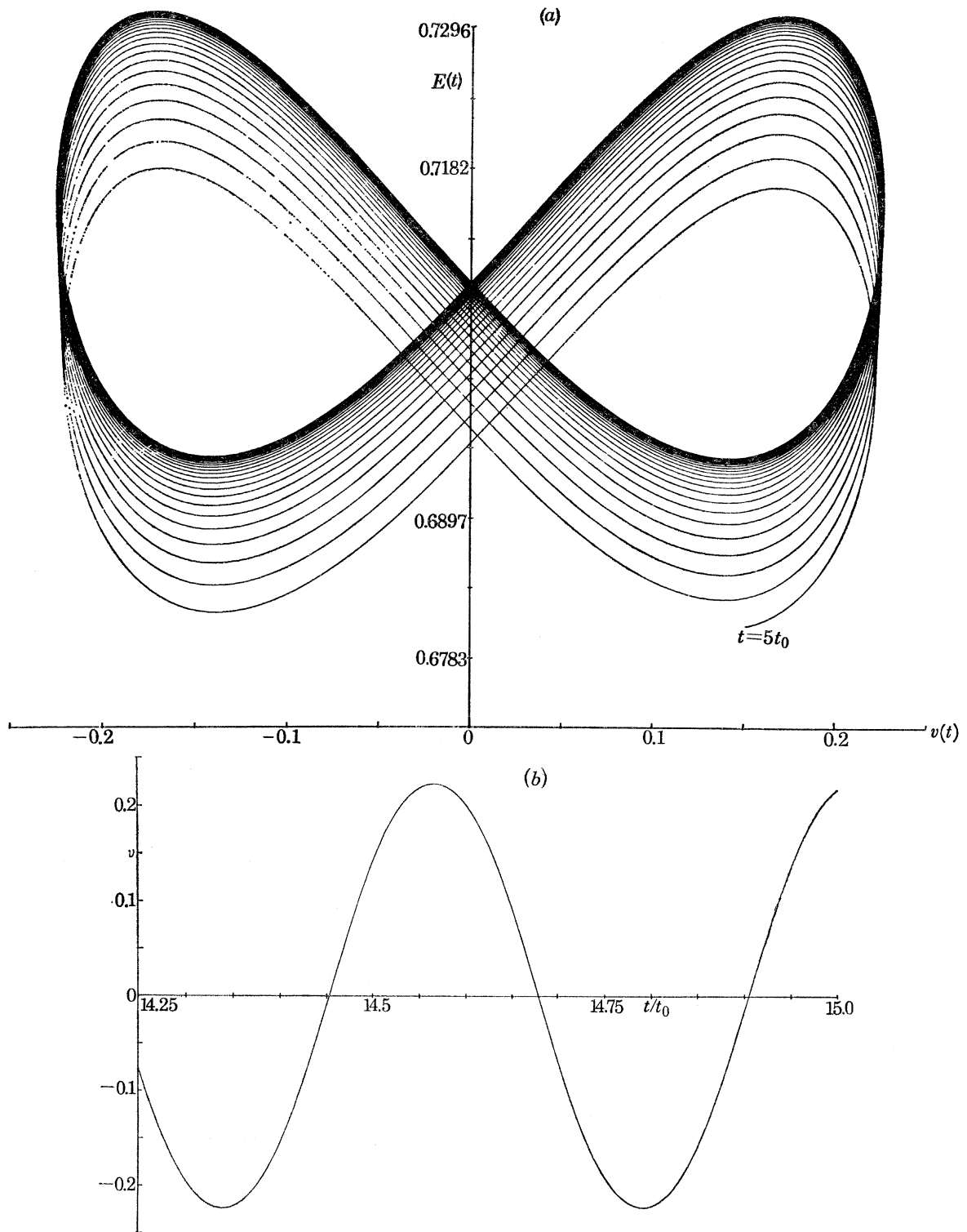


FIGURE 2(a). The magnetic energy 
$$E(t) = \frac{1}{2}M^2 \int_0^1 B_z^2 dz$$

is plotted against  $v(t)$  for the initial value problem (7.21) for the case  $\gamma = 0$ ,  $A = 1.548$ ,  $M^2 = 1.9$ . The plot begins at  $t = 5t_0$  and numerical integration with a time step  $\Delta t = t_0/1600$  is continued until  $t = 15t_0$ . (b) The corresponding plot of  $v$  against  $t$  during the period  $t = 14.25t_0$  to  $t = 15t_0$ .

slowly. However, the induction due to the large value of  $v$  becomes quite efficient. Consequently, both  $B_2^{(m)}$  and the magnetic energy  $E(t)$  increases. But, as  $v$  decreases from its maximum value,  $B_1^{(m)}$  is still decaying so that there is less  $B_1^{(m)}$  to convect. The decay continues until  $v$  approaches its largest negative value, where the process is repeated; but with the field components reversed.

The case  $\gamma \neq 0$  is also of some interest, since there is no reason to suppose that  $k$  is exactly equal to  $k_c$  in practice. Now, it has already been observed that, when  $\bar{M}^2$  is zero, one of the two rolls ultimately vanishes (see (7.7)). Therefore it is possible that the stabilizing influence of the magnetic field is not sufficient to overcome this tendency. If this is indeed the case, solutions obtained when  $\gamma = 0$  must be regarded with suspicion. Numerical integration of the case  $\gamma = 1$ ,  $\bar{M}^2 = 5$ ,  $A = 1.7$ , however, demonstrates the existence of such a hydromagnetic dynamo and the solution is illustrated in figure 3. Convergence to the final limit cycle of period  $0.8t_0$  is achieved after a time  $5t_0$  which is far quicker than the case  $\gamma = 0$ . It should be noted that in both the cases  $\gamma = 0$  and  $\gamma = 1$  the values of  $A$  chosen for the integrations were approximately the smallest values capable of sustaining the magnetic field. Several other calculations were, in fact, performed which indicated that as  $\gamma$  is increased so the value of  $A$  necessary to maintain the dynamo is also increased.

The behaviour of the solution in the limit cycle for the case  $\gamma = 1$  is discussed briefly. The oscillatory nature of  $v$  is qualitatively similar to its behaviour in the case  $\gamma = 0$ . There is, though, a tendency for it to approach  $-\frac{1}{2}$  and at this stage the field is clearly decaying quite rapidly. The loss of magnetic field at this end is compensated by relatively efficient dynamo action for values of  $v$  close to its maximum positive value. At this stage the mechanisms involved are presumably similar to the case  $\gamma = 0$ , when  $v$  is close to its maximum value.

(c) *Numerical results,  $N = 3$*

Though the simplest case to consider is that of two rolls, there is no reason to suppose that it would be attained in practice. For example, could a different combination of rolls provide a hydromagnetic dynamo with an even smaller value of  $A$ ? The results of this section suggest that this is indeed the case and pave the way for the solution discussed in § 8. The case  $N = 3$  is not general, of course, but it does highlight some new aspects of the hydromagnetic dynamo problem which are probably typical of the general case.

To keep the number of free parameters down to a minimum it is supposed that  $r(k)$  is the same for each roll. As in § 7*b*, it is necessary to consider the magnetic induction equation (7.19), but this time

$$M_{ij} = \begin{bmatrix} +\frac{1}{2}\sqrt{3}(q_2 - q_3) & -\frac{3}{2}(q_2 + q_3) \\ \frac{1}{2} + \frac{3}{2}q_1 & \frac{1}{2}\sqrt{3}(q_3 - q_2) \end{bmatrix}, \quad (7.22)$$

where  $q_i = q(t, \mathbf{k}_i)$  and  $\mathbf{k}_i$  is defined by (7.4). The nature of  $\mathbf{k}_i$  implies certain symmetries for the quantities  $A(\mathbf{k}', \mathbf{k})$  defined by (5.13). In fact (7.1) depends only on one parameter  $A$  where

$$A(\mathbf{k}_i, \mathbf{k}_j) + A(-\mathbf{k}_i, \mathbf{k}_j) = -\frac{1}{2}\pi^2 A \sum_{l=1}^3 \epsilon_{lij}. \quad (7.23)$$

Hence equation (7.1) reduces to the two equations

$$\frac{1}{\pi^2} \frac{dq_2}{dt} + \{(A + 4\sqrt{3}\bar{M}^2 \mathcal{L}(B_1 B_2)) q_3 - (A + 3\bar{M}^2 \mathcal{L}(B_2^2 - B_1^2) - 2\sqrt{3}\bar{M}^2 \mathcal{L}(B_1 B_2)) q_1\} q_2 = 0, \quad (7.24 a)$$

$$\frac{1}{\pi^2} \frac{dq_3}{dt} + \{(A - 3\bar{M}^2 \mathcal{L}(B_2^2 - B_1^2) - 2\sqrt{3}\bar{M}^2 \mathcal{L}(B_1 B_2)) q_1 - (A + 4\sqrt{3}\bar{M}^2 \mathcal{L}(B_1 B_2)) q_2\} q_3 = 0, \quad (7.24 b)$$

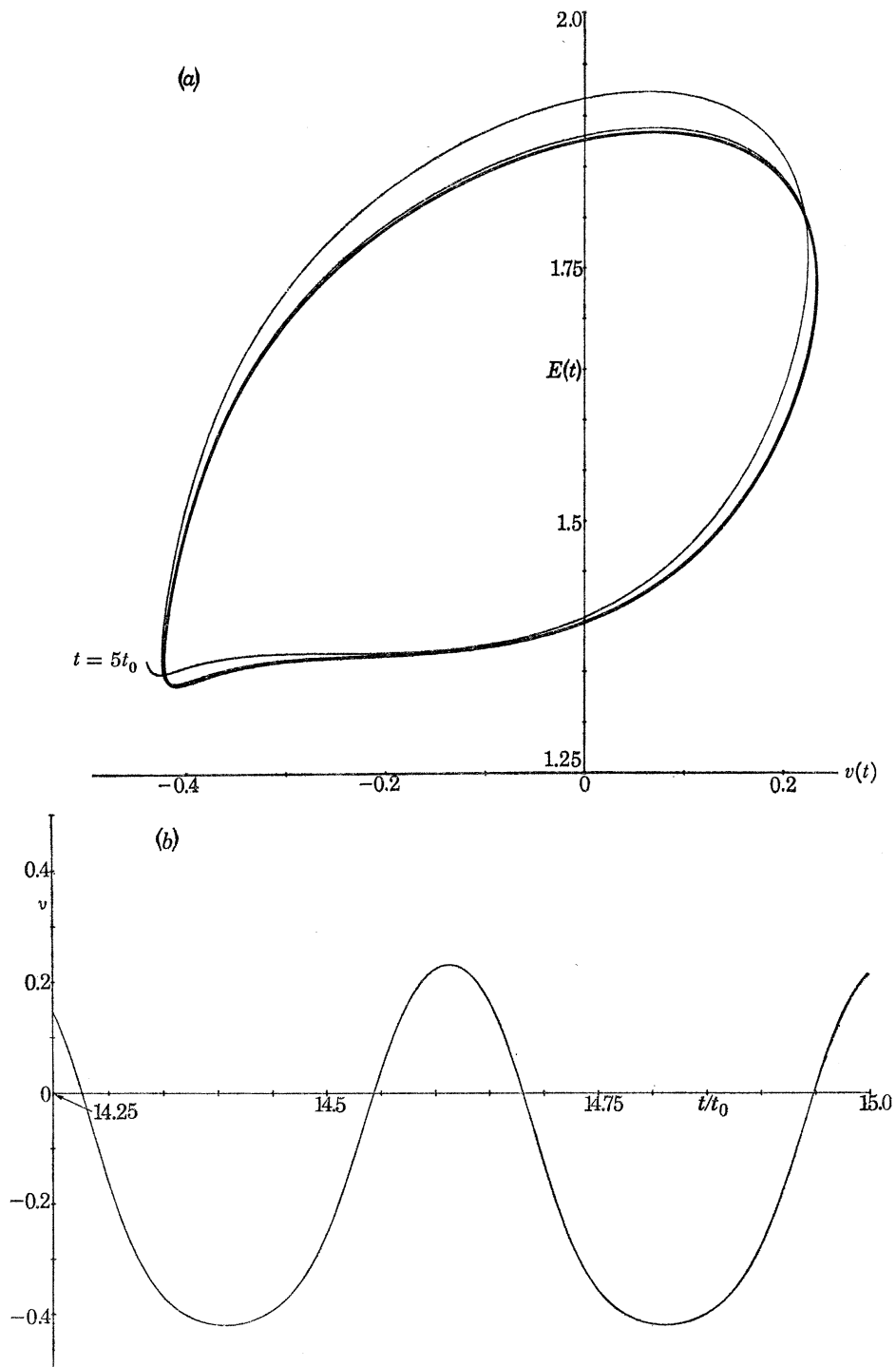


FIGURE 3. As for figure 1, for the case  $\gamma = 1$ ,  $\Lambda = 1.7$ ,  $\bar{M}^2 = 5$ .

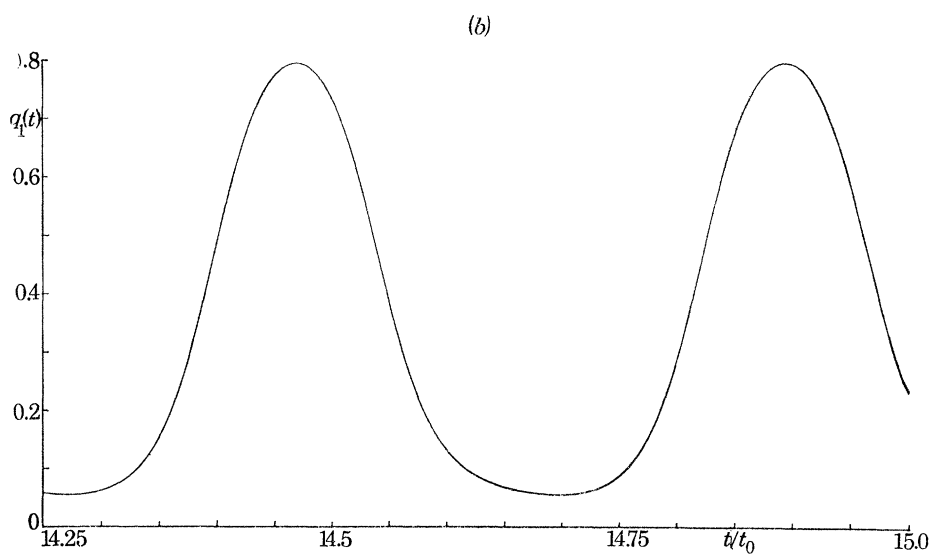
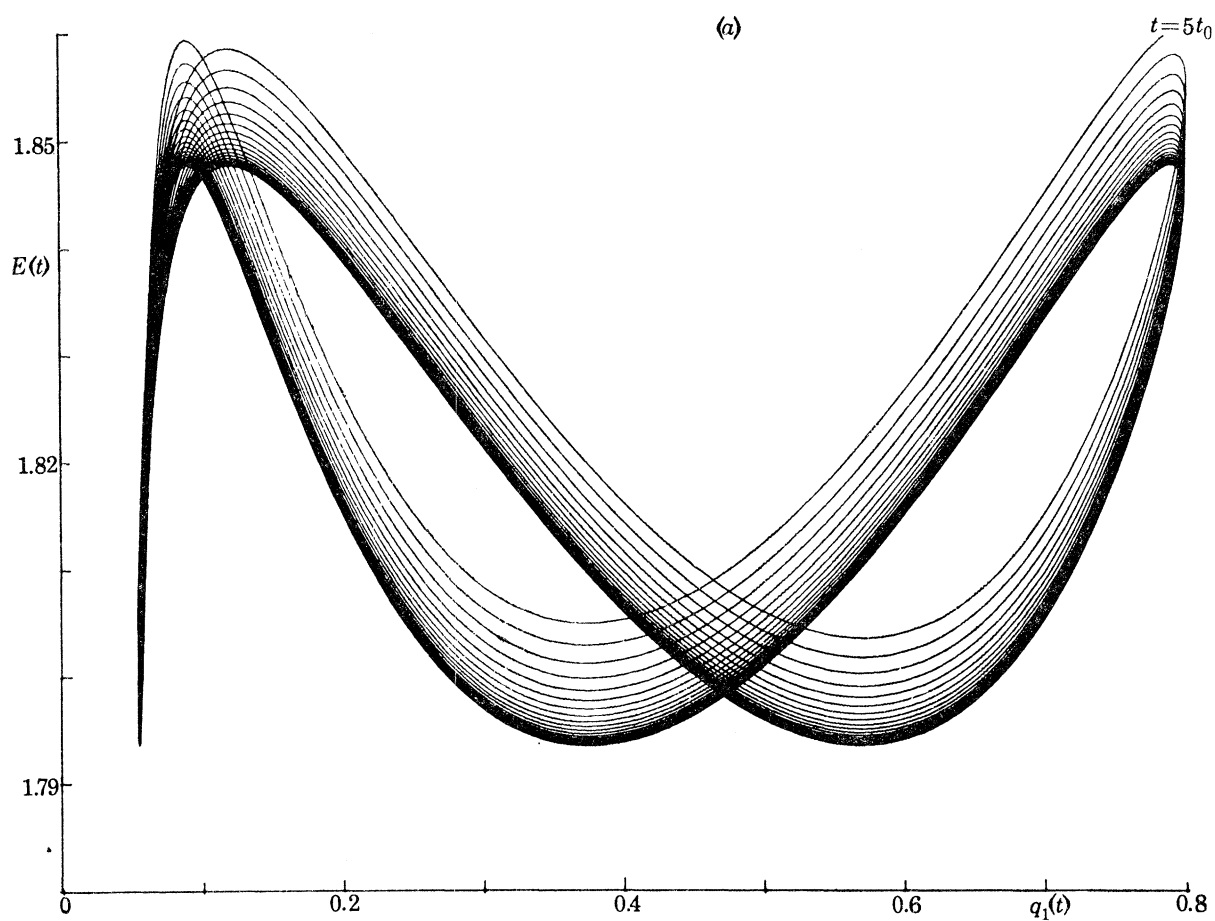
where

$$q_1 + q_2 + q_3 = 1 \quad (7.25)$$

and  $\mathcal{L}(B_i B_j)$  is defined by (7.20).

When the magnetic field vanishes (7.24) can be integrated once giving

$$q_1 q_2 q_3 = \frac{1}{2^7} (1 - \xi) \quad (0 \leq \xi \leq 1), \quad (7.26)$$

FIGURE 4 (*a* and *b*). For legend see following page.

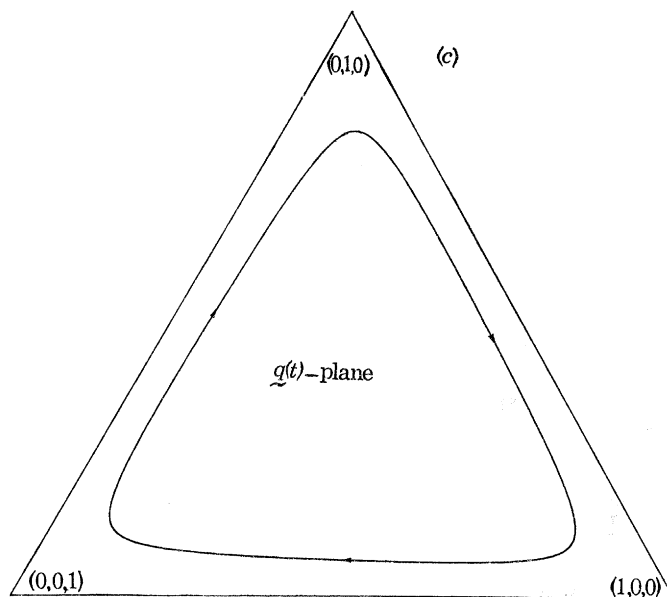


FIGURE 4(a). The magnetic energy  $E(t)$  is plotted against  $q_1(t)$  for the case  $A = 0$ ,  $\Lambda = 1.45$ ,  $\bar{M}^2 = 1.0$ . The plot begins at  $t = 5t_0$  and numerical integration with a time step  $\Delta t = t_0/1600$  is continued until  $t = 15t_0$ . (b) The corresponding plot of  $v$  against  $t$  during the period  $t = 14.25t_0$  to  $t = 15t_0$ . (c) The locus of the point  $(q_1, q_2, q_3)$  during the same period. Note the symmetries described in §7 and the beginning of §8.

where  $\xi$  is a constant. Consequently the orbit of  $\mathbf{q} = (q_1, q_2, q_3)$  lying on the plane defined by (7.25) is closed and similar to that illustrated in figure 4c. The position on the orbit at time  $t$  is most conveniently obtained by consideration of the quantity

$$q_1 q_2 + q_2 q_3 + q_3 q_1 = \frac{1}{3}[1 - \eta(t)]. \quad (7.27)$$

Thus  $q_1, q_2$  and  $q_3$  are the three roots of the cubic

$$q^3 - q^2 + \frac{1}{3}(1 - \eta)q - \frac{1}{27}(1 - \xi) = 0, \quad (7.28)$$

where  $\eta(t)$  satisfies the equation

$$(3/A^2\pi^4) (d\eta/dt)^2 = 4\eta^3 - (\xi - 3\eta)^2, \quad (7.29)$$

which has solutions in terms of elliptic integrals. Evidently an orbit corresponds to the trivial solution  $v = \text{constant}$  for the case of two rolls, when  $\gamma = \bar{M}^2 = 0$ . The solution (7.26) is an interesting result in the context of non-magnetic Bénard convection. First it shows that the hexagonal cell  $q = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$  remains steady. Secondly, though no other  $\mathbf{q}$  is stationary, the existence of closed orbits shows that the motion represented by the superposition of three rolls interacting at an angle of  $\frac{1}{3}\pi$  does not degenerate into a single roll.

The case  $A = 0$ ,  $\Lambda = 1.45$ ,  $\bar{M}^2 = 10$  is considered in detail. Since the dynamo fails with the initial data (7.21a) the initial condition  $B_1(0) + iB_2(0) = \Phi^*(z)$  is used instead. Numerical results with  $\Delta t = t_0/1600$ , are illustrated in figure 4. Figure 4a indicates that the solution approaches a limit cycle with period approximately  $0.9t_0$ , while figures 4b, c illustrate the dynamic behaviour in this limit cycle as measured by  $\mathbf{q}$ . It is interesting to note that in this cycle the point  $\mathbf{q}$  is never close to  $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ : instead, it keeps close to the boundary triangle. Moreover,  $\mathbf{q}$  spends most of its time in the vicinity of the corners and it is here that the magnetic energy is intensified (see figure 4a). The most natural symmetric plot of the magnetic energy is in a 3-dimensional

space formed by the  $\mathbf{q}$ -plane and the  $E$ -direction. Clearly the asymmetric character of figure 3 is misleading and is best regarded as a projection of the symmetric 3-dimensional configuration. An interpretation of the physical mechanisms involved is similar to that given in §7*b* for the case  $\gamma = 0$ .

The cases  $A = \pm 5$ ,  $A = 1.53$ ,  $\bar{M}^2 = 15$  were considered also. Limit cycles were approached for which the magnetic field displayed the same qualitative features as the case  $A = 0$ , while the magnetic energy  $E(t)$  fluctuated between 1.663 and 1.717 for the case  $A = 5$  and 4.063 and 4.196 for the case  $A = -5$ .

### 8. CONTINUOUS MODAL ANALYSIS

Though it is consistent with equations (3.21) and (7.1) to consider only a discrete set of modes, it is somewhat unsatisfactory to treat the stability of the system on this basis. Indeed, if the solutions are regarded to be the ultimate response of the system to an arbitrary initial disturbance, all modes with  $|\mathbf{k}| = k_c$  must be considered, together with modes in the neighbourhood of this circle. The full problem is evidently a formidable undertaking and is not attempted. Instead a class of periodic solutions, which can be represented by a continuous distribution of modes restricted to the circle  $|\mathbf{k}| = k_c$ , is investigated. As in the previous section the restriction on  $k$  isolates the influence of nonlinear effects and the Lorentz force in (7.1), since the term involving  $r(k)$  vanishes.

The solutions are suggested by the case  $N = 3$ , considered in §7*c*, which is hopefully typical of cases with  $N > 3$ . Obvious features of the solution are the negative frequency  $\omega$  and the property

$$q_3(t + 4\pi/3\omega) = q_2(t + 2\pi/3\omega) = q_1(t), \quad (8.1)$$

in the final limit cycle. Evidently the corresponding form for the continuous case is

$$q(t, \mathbf{k}(\theta)) = Q(\theta - \omega t), \quad (8.2a)$$

where

$$\mathbf{k}(\theta) = k(\cos \theta, \sin \theta). \quad (8.2b)$$

For the discrete case  $N = 3$ , a Fourier decomposition of the time dependent magnetic field includes all harmonics  $e^{in\omega t}$ , where  $n$  is an integer. However the magnetic energy is characterized by the property

$$E(t + 4\pi/3\omega) = E(t + 2\pi/3\omega) = E(t) \quad (8.3)$$

as is readily seen from the plot of  $E$  against  $q_1$  given in figure 4*a*. Thus the argument, which leads from (8.1) to (8.2), when applied to  $E(t)$ , implies that the magnetic energy is constant. Consequently the magnetic field is likely to be represented by a single harmonic, say

$$\mathbf{B}_h(z, t) = B_{\parallel}(z) \mathbf{k}(\omega t) + B_{\perp}(z) \mathbf{i}_z \times \mathbf{k}(\omega t), \quad (8.4)$$

for which the magnetic energy

$$E = \frac{1}{2} \bar{M}^2 \int_0^1 (B_{\parallel}^2(z) + B_{\perp}^2(z)) dz \quad (8.5)$$

is constant. It is a simple matter to show that in a frame rotating with angular velocity  $\omega$  both the magnetic field and velocity are steady but, of course, part of the steady motion is a solid body rotation with angular velocity  $-\omega$ . In spite of the plausibility arguments, it is remarkable that (8.2) and (8.4) are consistent with the governing equations and fortuitous that the resulting equations admit solutions!

The stability of the solutions is only investigated in the limit  $\bar{M}^2 \rightarrow 0$ , for which an analysis similar to that of §7*a* shows that the solutions are unstable. The arguments of the previous paragraph, however, do suggest that some of the solutions with finite  $\bar{M}^2$  are actually attained in practice as the limit cycle to a wide class of initial disturbances.

Substitution of (8.2) into (3.22) leads to the identity

$$M_{ij} B_j = \left[ \int_0^{2\pi} k^{-2} Q(\phi) \mathbf{i}_z \times \mathbf{k}(\theta) (\mathbf{k}(\theta) \cdot \mathbf{B}) d\phi \right]_i, \quad (8.6a)$$

where 
$$\phi = \theta - \omega t. \quad (8.6b)$$

Thus, since 
$$\mathbf{k}(\theta) \cdot \mathbf{k}(\omega t) = k^2 \cos \phi, \quad \mathbf{k}(\theta) \cdot \mathbf{i}_z \times \mathbf{k}(\omega t) = k^2 \sin \phi, \quad (8.7)$$

equation (3.21) reduces to

$$\omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} B_{\parallel} \\ B_{\perp} \end{bmatrix} + 2\pi A \begin{bmatrix} -Y & X-1 \\ X+1 & Y \end{bmatrix} \frac{\partial}{\partial z} \left( \sin 2\pi z \begin{bmatrix} B_{\parallel} \\ B_{\perp} \end{bmatrix} \right) - \frac{\partial^2}{\partial z^2} \begin{bmatrix} B_{\parallel} \\ B_{\perp} \end{bmatrix} = 0, \quad (8.8a)$$

where 
$$\int_0^{2\pi} Q(\phi) d\phi = 2, \quad X + iY = \frac{1}{2} \int_0^{2\pi} e^{i2\phi} Q(\phi) d\phi. \quad (8.8b)$$

Similarly, it is evident that, with the substitution of (8.2) into (5.13*a*),  $A(\mathbf{k}', \mathbf{k})$  becomes a function of  $\phi' - \phi$  alone, say

$$A(\mathbf{k}(\theta'), \mathbf{k}(\theta)) = \pi^2 a(\phi' - \phi), \quad (8.9)$$

while the contribution to the Lorentz force is

$$m(t, \mathbf{k}(\theta)) = 4\pi^2 \bar{M}^2 \mathcal{L}(B_{\parallel} \cos \phi + B_{\perp} \sin \phi)^2. \quad (8.10)$$

Hence the system of equations (7.1) reduces to the single equation

$$-\frac{\omega}{\pi^2} \frac{dQ}{d\phi} + 2 \left[ \int_0^{2\pi} a(\phi' - \phi) Q(\phi') d\phi' \right] Q - 2\bar{M}^2 \sin 2(\phi - \phi_0) Q = 0, \quad (8.11a)$$

where 
$$2\mathcal{L}(B_{\perp} B_{\parallel}) + i\mathcal{L}(B_{\perp}^2 - B_{\parallel}^2) = e^{2i\phi_0}, \quad (8.11b)$$

and 
$$\int_0^{2\pi} Q(\phi) \sin 2(\phi - \phi_0) d\phi = 0. \quad (8.11c)$$

Note that  $\phi_0$  does not provide an additional degree of freedom, but is instead a manifestation of the arbitrary orientation of  $\mathbf{k}(\omega t)$  to the direction of maximum  $Q$  (say).

Since  $a(\phi)$  is in general a rather cumbersome function, attention is restricted, henceforth, to the case  $a = 0$ . The case may be justified formally in the limit  $\eta/\nu \rightarrow 0$ , since  $a(\phi) \rightarrow 0$  also. It may be anticipated, however, that the qualitative features of the solution for non-zero  $a(\phi)$  are similar. This point of view is supported by the numerical results of §7*c* for non-zero  $A$ , and the assumption isolates the interesting influence of the Lorentz force by suppressing other nonlinear effects.

Integration of (8.11) yields the solution

$$Q(\phi) = [\pi I_0(\pi^2 \bar{M}^2/\omega)]^{-1} \exp [(\pi^2 \bar{M}^2/\omega) \cos 2(\phi - \phi_0)], \quad (8.12)$$

where  $I_n$  is the  $n$ th-order Bessel function of imaginary argument. Hence the coefficients  $X$  and  $Y$  in (8.8) are defined by

$$X + iY = e^{2i\phi_0} I_1(\pi^2 \bar{M}^2/\omega) / I_0(\pi^2 \bar{M}^2/\omega) = e^{2i\phi_0} \mu(\pi^2 \bar{M}^2/\omega). \quad (8.13)$$

With the choice  $\phi_0 = 0$ ,  $Y$  vanishes and  $X = \mu$ . The resulting eigenvalue problem (8.8) is considered for various values of  $\Lambda$  and  $\mu$ . The solution only has meaning as a kinematic dynamo when (i)  $\omega$  is real, and the dynamical conditions are satisfied only when (ii)  $\mathcal{L}(B_{\perp}^2 - B_{\parallel}^2) = 0$ , (iii)  $2\mathcal{L}(B_{\perp} B_{\parallel}) = 1$  and  $\omega\mu$  is positive. Provided these conditions are met the equation

$$I_1(\xi) = \mu I_0(\xi), \quad (8.14)$$

is solved to determine  $\xi(\mu)$ . This, in turn, fixes the strength of the magnetic field as measured by the magnetic energy

$$E = (\omega/2\pi^2) \xi(\mu) \int_0^1 (B_{\parallel}^2 + B_{\perp}^2) dz. \quad (8.15)$$

Consequently, even though the problem is nonlinear, the nonlinearity is confined to the conditions (ii) and (iii) and the simple equation (8.14) which is solved after the dynamo is established! Herein lies the success of the method, as the linear eigenvalue problem is readily solved numerically.

As in §7 only the smallest odd mode is considered. The Fourier representation

$$(B_{\parallel}, B_{\perp}) = \sum_{n=0}^{\infty} (D_{2n}, D_{2n+1}) \cos(2n+1)\pi z, \quad (8.16)$$

is substituted into (8.8) and leads to the eigenvalue problem

$$\sum_{m=0}^{\infty} (N_{nm} - (\omega/\pi^2) \delta_{nm}) D_m = 0 \quad (n \geq 0), \quad (8.17a)$$

where

$$\left. \begin{aligned} -N_{2m, 2m+1} &= N_{2m+1, 2m} = (2m+1)^2 & (m \geq 0), \\ -N_{2m, 2m-2} &= N_{2m, 2m+2} = \Lambda(\mu+1)(2m+1) & (m \geq 1), \\ N_{2m+1, 2m-1} &= -N_{2m+1, 2m+3} = \Lambda(\mu-1)(2m+1) & (m \geq 1), \\ -N_{0,0} &= N_{0,2} = \Lambda(\mu+1), \\ N_{1,1} &= -N_{1,3} = \Lambda(\mu-1), \\ N_{m,n} &= 0 \quad \text{otherwise.} \end{aligned} \right\} \quad (8.17b)$$

Numerical calculations were performed with only the first ten harmonics so that after truncation  $N_{nm}$  is a  $20 \times 20$  matrix with non-zero elements restricted to five diagonals. For fixed  $\mu$ , all the eigenvalues  $\omega/\pi^2$  are pure imaginary when  $\Lambda = 0$ . As  $\mu$  is increased, the eigenvalues are grouped in complex conjugate pairs, until one pair coalesce and subsequently split into two distinct real eigenvalues. Moreover, a curve  $c$  is located in the  $(\Lambda, \mu)$  plane (see figure 5a) for which all the conditions (i), (ii) and (iii) hold and corresponds to the kinematic dynamo solution with the larger of the two frequencies. The values of the frequency and magnetic energy of the hydro-magnetic dynamo solution are plotted against  $\Lambda$  in figure 5b.

The degenerate steady solution  $\Lambda = \Lambda_{\text{crit}}$ ,  $\mu = 0$  marks the end of the curve  $c$ . It should be noted that the eigenvalues of the kinematic problem are repeated at this point and are complex in any neighbourhood (see §3 and appendix B). The stability of the hydromagnetic solutions near this point are readily treated, since  $\bar{M}^2$  is small, by the technique adopted in §7a. It is supposed that

$$\Lambda = \Lambda_{\text{crit}} + \bar{M}^2 \Lambda^{(2)} \quad (\bar{M}^2 \ll 1), \quad (8.18a)$$

$$q(t, \mathbf{k}) = 1/\pi + \bar{M}^2 q^{(2)}(t, \mathbf{k}) + O(\bar{M}^4). \quad (8.18b)$$



As before  $\phi$  is defined by (3.23*b*) and a solution is sought of the form (7.13). The order  $\bar{M}^2$  contributed to  $q$ , corresponding to (7.14), is

$$q^{(2)}(t, \mathbf{k}(\theta)) = \text{Re} [(a^2 \pi / i \omega_0) \mathcal{L}(\Phi^2) e^{2i(\omega_0 t - \theta)}], \quad (8.19)$$

and hence 
$$\alpha_{ij} = \delta_{ij} + \bar{M}^2 (\pi^2 / 2 \omega_0) \text{Re} \left\{ i a^2 \mathcal{L}(\Phi^2) \begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix} e^{2i \omega_0 t} \right\} + O(\bar{M}^4). \quad (8.20)$$

The arguments leading to (7.15) are applied directly to (3.23*a*) and lead to a similar equation governing the amplitude  $a(\tau)$ . The corresponding periodic solution  $X = \frac{1}{2} X_0$  (cf. (7.17)) is represented by a point on the curve  $c$  and the above analysis indicates that it is unstable.

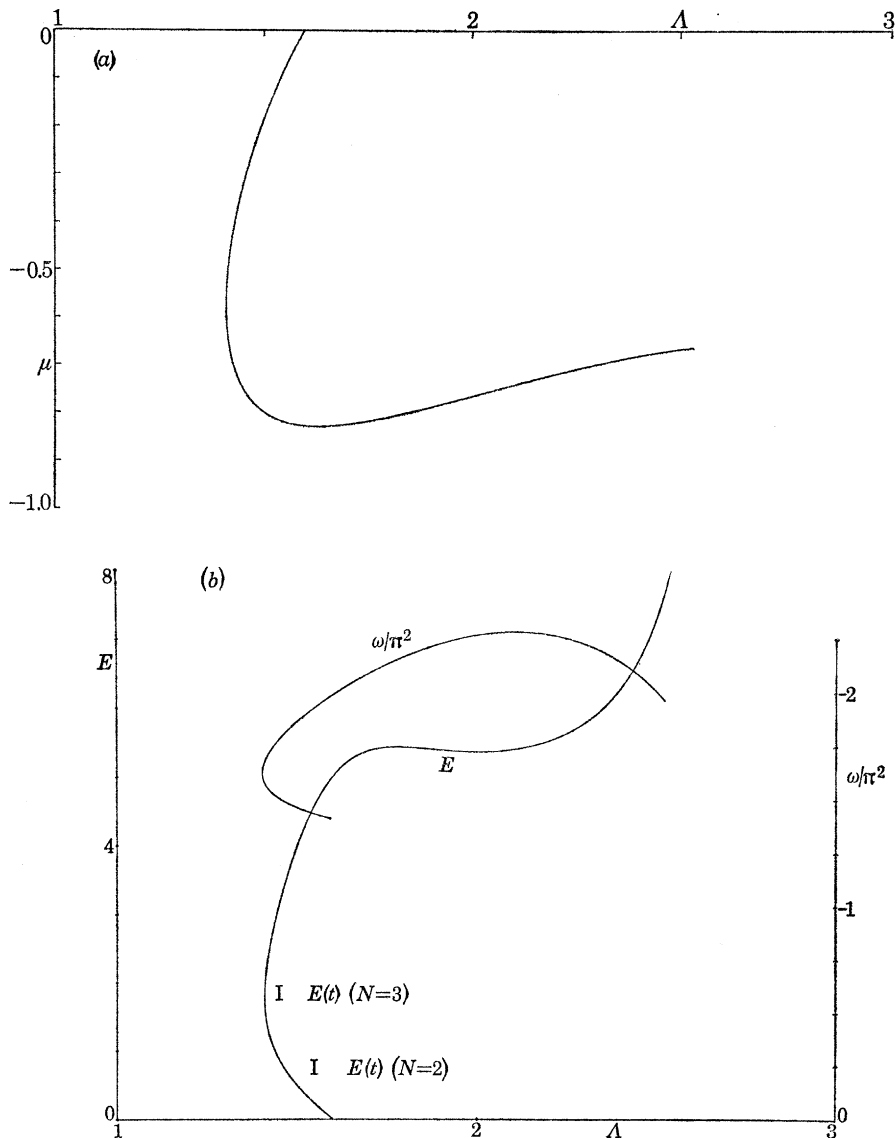


FIGURE 5(a). Points on the curve  $c$  in the  $A, \mu$ -plane correspond to solutions of the eigenvalue problem (8.8) for which (i)  $\omega$  is real, (ii)  $\mathcal{L}(B^2 - B^2) = 0$ , (iii)  $2\mathcal{L}(B_1 B_1) = 1$  and  $\omega\mu \geq 0$ . (b) The frequency  $\omega$  and the magnetic energy  $E$  for hydromagnetic dynamos corresponding to points on  $c$  is given as a plot against  $A$ . The lines  $A = 1.45$ ,  $1.79 \leq E \leq 1.85$  and  $A = 1.55$ ,  $0.69 \leq E \leq 0.73$  summarize the results for the case of three and two rolls already illustrated in figures 4 and 2 respectively.

There is, however, no reason to suppose that all solutions defined by points on  $c$  are unstable. For example, it may be postulated that solutions are stable after some point on the curve is reached, say at the minimum value of  $\Lambda$ , located approximately at the point

$$\Lambda_{\min} = 1.408, \quad \mu = -0.58 \quad (E = 1.8). \quad (8.21)$$

Such a hypothesis is suggested by the existence of the stable hydromagnetic dynamos with finite magnetic energy  $E(t)$  considered in §§7*b*, *c*. One final point; it is reasonable to speculate that no hydromagnetic dynamos are possible with  $\Lambda < \Lambda_{\min}$ . If this is the case, there are clearly kinematic dynamos with even smaller  $\Lambda$  (e.g.  $\Lambda = \Lambda_{\text{rot}}$ ). However, the value  $\Lambda_{\min}$  is still substantially less than  $\Lambda_{\text{crit}}$  obtained for kinematic dynamos based on steady motion.

## 9. CONCLUDING REMARKS

The analysis of the paper has established the existence of a hydromagnetic dynamo subject to two important factors. First, the maintenance of a constant temperature difference across the layer results in buoyancy forces that drive motions capable of sustaining a magnetic field. The hydromagnetic dynamo is therefore, natural in the sense that an artificial force field is not invoked to maintain suitable asymmetric motions. Secondly, some of the solutions are stable. The stability of the hydromagnetic dynamo is clearly indicated by the numerical integration of initial value problems in §§7*b*, *c*, where the solutions rapidly approach limit cycles. Though this result is only established numerically, comparison with the periodic solutions of §8 suggests that some (but not all) of the analytic solutions involving the continuous distribution of modes are stable. Perhaps it should be pointed out that steady solutions could have been obtained analytically based on the kinematic dynamo solution (3.26). These solutions were never investigated on the grounds that they are unstable. Specifically, there is always part of the magnetic field that grows exponentially when  $\alpha\Lambda$  is a zero of the Bessel function  $J_0$ .

Since the influence of the Lorentz force is to relax the constraining influence of rotation (see, for example, Malkus 1959 and Eltayeb 1972), the role of the magnetic field in stabilizing the hydromagnetic dynamo is curious. It would be expected that rather than stabilize the system an increase in the magnetic field would in turn tend to increase the amplitude of the motion. This does not happen for the models considered because the magnetic field is so weak. More precisely, the general question concerning the intensification of the kinetic energy of the fluid does not arise, as this is fixed by other considerations. Hence the Lorentz force provides only a selection mechanism: any general tendency to amplify motions is balanced by buoyancy forces which result from a modification to the mean vertical temperature as measured by  $\Theta^{(2)}(t)$ . Consequently thermal convection stabilizes the dynamical system†, as the results of §6 indicate. In brief, the gross features of the finite amplitude dynamics are dominated by nonlinear convection of buoyancy and it is only the fine structure which depends on the Lorentz force.

The arguments of the previous paragraph suggest that the hydromagnetic dynamos considered in this paper are special. Indeed it is probably typical for the magnetic field to adjust in such a way that Lorentz forces are comparable in magnitude with Coriolis forces. For, if this is the case, the Taylor number required for motion to occur is smaller than  $T^{\frac{2}{3}}\tilde{R}^{(0)}(k_c)$  by orders of magnitude. Evidently analysis of this problem along the lines of the present paper is a far more difficult undertaking and is not attempted here. The present paper, therefore, does not claim to describe

† The notion of dynamic stability should not be confused with the stability of the dynamo.

the typical convection driven dynamo in a rotating system but rather a special model which is amenable to substantial analytic treatment.

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#### APPENDIX A

Referred to the cartesian axes adopted in this paper, Busse (1973) considers rolls whose axes are aligned to (say) the  $y$ -direction. Superimposed on the rolls is a shear flow, also in the  $y$ -direction, whose magnitude depends only on the vertical coordinate  $z$ . Dynamically the two motions are decoupled: neither motion influences the other. Since the flow is only a function of  $x$  and  $z$  solutions of the magnetic induction equation can be sought of the form

$$\mathbf{b} = \tilde{\mathbf{b}}(x, z) \exp\{\sigma t + i\beta y\}.$$

When  $\beta$  is given, this leads to a linear eigenvalue problem for  $\sigma$ . For a suitable choice of velocity and small  $\beta$ , Busse shows that

$$\sigma = i\beta\sigma^{(1)} + \beta^2\sigma^{(2)} + \dots \quad (\beta \ll 1),$$

where  $\sigma^{(1)}$  and  $\sigma^{(2)}$  are real. Evidently dynamo action is possible when  $\text{Re } \sigma^{(2)} \geq 0$ . Since the magnitude of the shear flow is given the condition  $\text{Re } \sigma^{(2)} = 0$  for periodic magnetic fields fixes the strength of the rolls so that the flow is determined completely. At the same time the magnetic field is determined up to an arbitrary constant, as measured (say) by the mean magnetic energy. Finally consideration of the equations of motion and heat conduction, taking account of non-linear effects, determines a relation between the magnetic energy and Rayleigh number.

The expansion procedure hinges on the assumption that  $\beta$  is small. Though small values of  $\beta$  may correspond to the most easily excited dynamos, the choice is arbitrary. Consequently only a restricted class of solutions of the magnetic induction equation are considered. Indeed it appears likely, if higher terms in the expansion of  $\sigma$  are considered that the minimum value of  $\beta$  for which dynamo action is possible is obtained as  $\beta \rightarrow 0$ ! The case  $\beta = 0$  is ruled out, of course, as part of the magnetic field in the limit  $\beta \rightarrow 0$  is uniform and the model is no longer a dynamo. In the present paper, the second *ad hoc* length scale  $\beta^{-1}L (\gg L)$  is replaced by the length scale  $T^{-1/2}L (\ll L)$  imposed naturally by the rapid rotation of the system. The distinction is important as the success of the analytic techniques in both models stem from multiple length scale procedures.

The second major difference is in the choice of motion. Ignoring the shear velocity for the moment, rolls with axes in any horizontal direction are possible at the onset of instability, and moreover any superposition of these rolls is acceptable on the basis of linear theory. The amplitude of each roll is arbitrary and, consequently, to lowest order there are  $N$  undetermined parameters, where  $N$  is the number of independent rolls. When just one roll combined with a given shear flow is considered, there is only one free parameter which, as already noted, transpires to be constant for all time. In the present paper, where  $N (\geq 2)$  rolls are considered, the amplitude of each roll varies on the diffusion time scale as a direct consequence of nonlinear interactions and the Lorentz force. Moreover, since the response of the flow to time-dependent Lorentz forces is itself periodic, magnetic fields are maintained for smaller values of the magnetic Reynolds number

(and hence Rayleigh number) than predicted for steady flow. Thus in many cases the very existence of the dynamo depends on the dynamic coupling of the Lorentz force. The situation has some similarity with the phenomena of subcritical instability when finite amplitude motions are possible for Rayleigh numbers smaller than their critical values.

TABLE 1. THE FOURIER COEFFICIENTS OF  $\Phi(z)$  AND  $\Phi^A(z)$ 

$m$	$\text{Re } A_m$	$\text{Im } A_m$	$\text{Re } A_m^A$	$\text{Im } A_m^A$
0	1.0	0.0	1.0	0.0
1	$1.275 \times 10^{-1}$	$-6.260 \times 10^{-1}$	$-4.251 \times 10^{-2}$	$2.087 \times 10^{-1}$
2	$-2.128 \times 10^{-1}$	$-5.747 \times 10^{-2}$	$-4.257 \times 10^{-2}$	$-1.149 \times 10^{-2}$
3	$-1.521 \times 10^{-2}$	$5.021 \times 10^{-2}$	$2.173 \times 10^{-3}$	$-7.173 \times 10^{-3}$
4	$9.099 \times 10^{-3}$	$2.936 \times 10^{-3}$	$1.011 \times 10^{-3}$	$3.263 \times 10^{-4}$
5	$4.502 \times 10^{-4}$	$-1.340 \times 10^{-3}$	$-4.093 \times 10^{-5}$	$1.218 \times 10^{-4}$
6	$-1.664 \times 10^{-4}$	$-5.746 \times 10^{-5}$	$-1.280 \times 10^{-5}$	$-4.420 \times 10^{-6}$
7	$-6.294 \times 10^{-6}$	$1.786 \times 10^{-5}$	$4.196 \times 10^{-7}$	$-1.191 \times 10^{-6}$
8	$1.689 \times 10^{-6}$	$6.044 \times 10^{-7}$	$9.935 \times 10^{-8}$	$3.556 \times 10^{-8}$
9	$5.170 \times 10^{-8}$	$-1.427 \times 10^{-7}$	$-2.721 \times 10^{-9}$	$7.511 \times 10^{-9}$

## APPENDIX B

Any solution of the equation

$$\frac{d^2\phi}{dz^2} - i2\pi A \frac{d}{dz} (\phi \sin 2\pi z) - \pi^2 s\phi = 0, \quad (\text{B1})$$

subject to the boundary condition  $d\phi/dz = 0$  on  $z = 0$  and  $1$  can be expressed in either the form

$$\sum_{m=0}^{\infty} A_m \cos(2m+1)\pi z \quad \text{or} \quad \sum_{m=1}^{\infty} B_m \cos 2m\pi z. \quad (\text{B2})$$

In order to obtain the eigenvalues  $\pi^2 s$  numerically, the series representations (B2) were assumed and the corresponding terms in (B1) were truncated at a finite number  $N$  (say). The problem was thus reduced to determine numerically the eigenvalues and eigenvectors of a complex tri-diagonal  $N \times N$  matrix. The eigenvalues of the matrix were determined with  $N = 60$ . However, the value of  $s_0$  obtained with  $N = 60$  and  $N = 10$  were identical to at least nine decimal places.

It was found that, when

$$A = A_{\text{min}} = 1.5974, \quad (\text{B3})$$

the first six eigenvalues are  $-i1.3936$ ,  $-2.270$ ,  $-7.584 - i0.2005$ ,  $-14.65$ ,  $-23.68 - i0.0032$ ,  $-34.69$ , where  $s_n$  is complex for  $n$  odd, and real for  $n$  even.

The adjoint equation governing  $\phi^A$  is

$$\frac{d^2\phi^A}{dz^2} + i2\pi A \sin 2\pi z \frac{d\phi^A}{dz} - \pi^2 s\phi^A = 0 \quad (\text{B4})$$

and the boundary conditions on  $\phi^A$  are that  $d\phi^A/dz = 0$  on  $z = 0, 1$ . In order to evaluate the integrals appearing in §7a it is necessary to compute  $\Phi$  and  $\Phi^A$ ; the eigenfunctions corresponding to

$$A = A_{\text{min}}, \quad s = -1.3936i. \quad (\text{B5})$$

The coefficients of their Fourier series are listed in the table. Using these results it is easy to show that

$$(\Phi^A, \Phi^*) = [0.7681 - i0.03952] (\Phi^A, \Phi), \quad (\text{B6})$$

$$\frac{1}{\pi^2} \left( \frac{d\Phi^A}{dz}, \frac{d\Phi}{dz} \right) = [2.0979 + i0.4323] (\Phi^A, \Phi), \quad (\text{B7})$$

$$\mathcal{L}(\Phi^2) = [0.6172 - i0.3232] - (k^6/\pi^2) [0.04693 + i0.1864], \quad (\text{B8})$$

$$= 0.5937 - i0.4163 \quad (k = k_c). \quad (\text{B9})$$

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